Section 10.2: Calculus With Parametric Equations

Review

Parametric equations are functional values for x and y coordinates

$x = f(t)$

$y = g(t)$

Questions

Verify that the following parameterizations are all for the same curve.

- $x = 2\cos(3t)$, $y = 2\sin(3t)$, $0 \le t \le \pi/3$
- $x = t$, $y = \sqrt{4 - t^2}$, $-2 \le t \le 2$
- $x = 2\sin(t)$, $y = 2\cos(t)$, $-\pi/2 \le t \le \pi/2$

Tangent lines

Questions

- If $y = x^2 + 1$, compute $\frac{dy}{dx}\,|_{x=1}$. What does this number represent?
- Consider the Tschirnhausen curve given by $y^2 = x^3 + 3x^2$.
  - Verify $(1, 2)$ is a point on this curve.
  - Use implicit differentiation to find an equation for the tangent line to this curve at $(1, 2)$.

Slopes of curves defined by parametric equations

Slope is rise over run. Infinitesimally, this is $\frac{dy}{dx}$. For parametric equations, the chain rule implies $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Questions

- If $x = 2t^2 + 1$, $y = 3t^3 - 4$, find $\frac{dy}{dx}$.

- $x = 2(t - \sin(t))$, $y = 2(1 - \cos(t))$ are parametric equations for a cycloid. Find an equation for the tangent line to this curve where $t = \pi/6$.
- $x = \cos(t)$, $y = \sin(2t)$ are parametric equations for a curve.
  - Find all points on this curve that have horizontal tangents.
  - Find all points on this curve that have vertical tangents.
  - Is (0, 0) on this curve?

Concavity

The curve $y = f(x)$ is concave up wherever $\frac{d^2y}{dx^2} > 0$, and it is concave down where $\frac{d^2y}{dx^2} < 0$. For parametric equations using the chain rule

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)}{\frac{dx}{dt}}$$

Questions

- If $x = 2t^2 + 1$, $y = 3t^3 - 4$, find $\frac{d^2y}{dx^2}$.
- $x = 2(t - \sin(t))$, $y = 2(1 - \cos(t))$ are parametric equations for a cycloid. Is the cycloid concave up or down at $t = \pi/6$?
- $x = \cos(t)$, $y = \sin(2t)$ are parametric equations for a curve. For which values of t in $[0, 2\pi]$ is the curve concave up?

Curve length

Given parametric equations for a curve

$$x = x(t),\ y = y(t),\ \ a \le t \le b$$

the length of the curve is

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

Question

Find the length of the curve $x = 3\cos(t)$, $y = 3\sin(t)$, $0 \le t \le \pi$.

Note: Only on rare occasions like the above example, can these integrals be evaluated exactly. Because of the square root, usually these length integrals must be integrated using numerical techniques.

Questions

- The cycloid $x = 2(t - \sin(t))$, $y = 2(1 - \cos(t))$ lies above the x-axis. Determine the values of parameter t where it touches the x-axis.

- Find the length of one arch of the above cycloid.

Questions

Consider the parametrically defined curve

$x = \cos(3\,t)$

$y = \sin(2\,t)$

- This curve starts repeating. What is its period?
- Set up the integral to find the length of this curve.

Why the formula works

For a given curve defined by parametric equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, divide $[a, b]$ into n subintervals of equal length $\Delta t = \frac{b-a}{n}$,

$t_0 = a,\ t_1 = a + \Delta t,\ t_2 = a + 2\,\Delta t,\ t_3 = a + 3\,\Delta t,\ \ldots,\ t_n = a + n\,\Delta t = b$

Approximate the curve with the line segments from $(x(t_i), y(t_i))$ to $(x(t_{i+1}), y(t_{i+1}))$.

Question

What is the length of the line segment from point $(x(t_i), y(t_i))$ to $(x(t_{i+1}), y(t_{i+1}))$?

An approximation for the length of the curve is the sum of the lengths of these individual line segments.

$$\text{Length} \approx \sqrt{(x(t_0) - x(t_1))^2 + (y(t_0) - y(t_1))^2} + \sqrt{(x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2} + \ldots + \sqrt{(x(t_{n-1}) - x(t_n))^2 + (y(t_{n-1}) - y(t_n))^2}$$

$$= \sum_{i=1}^{n} \sqrt{(x(t_{i-1}) - x(t_i))^2 + (y(t_{i-1}) - y(t_i))^2}$$

The squared quantities are differences. Make them into difference quotients.

$$= \sum_{i=1}^{n} \sqrt{\frac{(x(t_{i-1})-x(t_i))^2}{\Delta t^2}\,\Delta t^2 + \frac{(y(t_{i-1})-y(t_i))^2}{\Delta t^2}\,\Delta t^2}$$

$$= \sum_{i=1}^{n} \sqrt{\left(\frac{(x(t_{i-1})-x(t_i))^2}{\Delta t^2} + \frac{(y(t_{i-1})-y(t_i))^2}{\Delta t^2}\right)\Delta t^2}$$

$$= \sum_{i=1}^{n} \sqrt{\left(\frac{x(t_{i-1})-x(t_i)}{\Delta t}\right)^2 + \left(\frac{y(t_{i-1})-y(t_i)}{\Delta t}\right)^2}\ \Delta t$$

For large n this becomes a better approximation. In the limit as $n \to \infty$ this expression goes to

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$