Section 11.4: Comparison Test

Review

Integral test

Given the infinite series $\sum_{k=a}^{\infty} a_k$, if there is an integrable function $f(x)$ such that

- $f(n) = a_n$ for $n \geq a$,
- $f(x) \geq 0$ for $x \geq a$,
- $f(x)$ is a decreasing function for $x \geq a$.

then the infinite series $\sum_{k=a}^{\infty} a_k$ converges if and only if the improper integral $\int_a^{\infty} f(x)\,dx$ converges.

Question

For which values of p does the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge and for which values of p does it diverge?

Integral test error estimate

Partial sum approximations

If you know $\sum_{k=n_0}^{\infty} a_k$ converges then you can approximate it with its partial sums.

Integral test error estimate

If the integral test implies that $\sum_{k=1}^{\infty} a_k$ converges using the improper integral $\int_1^{\infty} f(x)\,dx$, then the error in using the n^{th} partial sum to approximate $\sum_{k=1}^{\infty} a_k$ satisfies

$$\text{error} = \left| \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k \right| = \sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} f(x)\,dx$$

Questions

- Give an estimate of the error in using the 10^{th} partial sum of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ to approximate it.

- Find a value of n for which you know the finite sum $\sum_{k=1}^{n} \frac{1}{k^5}$ will approximate the infinite sum $\sum_{k=1}^{\infty} \frac{1}{k^5}$ with error less than 0.0001.

Comparison tests generalities

Notes:

Just as the Integral Test can only be used on sums with nonnegative terms, so too the comparison tests we consider.
Technically we can say that this must be true only “**eventually**”: you can always peel off the first few terms of a series (a finite chunk), and consider the series as a sum of that finite chunk and a new series (which satisfies the conditions of the integral test, etc.).

What to expect

The comparison tests allow us to compare a new series with those we are already familiar with. If the comparison is good, then we can transfer what we know about **convergence** of the old series to the new one.

Questions

- p-series: For which values of p does $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge and for which does it diverge?
- Geometric series: For which values of r does $\sum_{k=1}^{\infty} a\, r^k$ converge and for which does it diverge?

Limit comparison test

Questions

Which p-series or geometric series is the given series similar to? Do you think the given series converges or diverges? How do we analyze them?

- $\sum_{k=1}^{\infty} \frac{1}{k^2+4\,k^3}$
- $\sum_{k=1}^{\infty} \frac{3^k}{2^k+k}$

Limit Comparison Test

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where L is a nonzero number, then $\sum_{k}^{\infty} b_k$ converges if and only if $\sum_{k}^{\infty} a_k$ converges.

Idea behind Limit Comparison Test

Given a complicated series, ignore all but the most significant term in the numerator and the significant term in the denominator where significant is in terms of large values of the index. This may suggest a new (and simpler) series, with which we can compare the give series.

Questions

Determine which series converge and which diverge.

- $\sum_{k=1}^{\infty} \frac{3k-2}{k^2+4}$
- $\sum_{k=0}^{\infty} \frac{10k+1}{k^3+5k^2-k+4}$
- $\sum_{k=0}^{\infty} \frac{2^k+\sin(k)+10}{5^k}$
- $\sum_{k=1}^{\infty} \frac{5}{k+2^k}$
- $\sum_{k=1}^{\infty} \frac{1+k^2+k^3+3^k}{2^k+k^4}$

Comparison test (based on inequalities)

Questions

- Which is bigger: $\frac{1}{k^2}$ or $\frac{1}{k^2+4k}$?
- Does $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converge or diverge?
- What do you think is true about $\sum_{k=1}^{\infty} \frac{1}{k^2+4k}$?

Questions

- Which is bigger: $\frac{1}{k}$ or $\frac{1}{k-1}$?
- Does $\sum_{k=2}^{\infty} \frac{1}{k}$ converge or diverge?
- What do you think is true about $\sum_{k=2}^{\infty} \frac{1}{k-1}$?

Comparison test

If you want to know whether $\sum_{k}^{\infty} a_k$ converges or not and you find a simpler series $\sum_{k}^{\infty} b_k$ whose convergence/divergence you know, then

- If $0 \leq a_k \leq b_k$ and $\sum_k^\infty b_k$ converges, then $\sum_k^\infty a_k$ must converge as well.
- If $a_k \geq b_k \geq 0$ and $\sum_k^\infty b_k$ diverges, then $\sum_k^\infty a_k$ must diverge as well.

Comparison test error estimate

If $0 \leq a_k \leq b_k$, $\sum_k^\infty b_k$ converges, and error $= \left|\sum_k^\infty b_k - \sum_k^n b_k\right| = \sum_{k=n+1}^\infty b_k \leq L$, then

$$\text{error} = \left|\sum_k^\infty a_k - \sum_k^n a_k\right| = \sum_{k=n+1}^\infty a_k \leq \sum_{k=n+1}^\infty b_k \leq L$$

Questions

Consider the series $\sum_{k=1}^\infty \frac{1}{k\,2^k}$.

- Compare this series to a geometric series to show it converges.
- What is an error estimate in using $\sum_{k=1}^9 \frac{1}{k\,2^k}$ to approximate $\sum_{k=1}^\infty \frac{1}{k\,2^k}$?

Questions

Consider the series $\sum_{k=1}^\infty \frac{1}{k^3-2\,k^2+4\,k+7}$.

- Compare this series to a p-series to show it converges.
- What is an error estimate in using $\sum_{k=1}^{10} \frac{1}{k^3-2\,k^2+4\,k+7}$ to approximate $\sum_{k=1}^\infty \frac{1}{k^3-2\,k^2+4\,k+7}$?