Section 12.2: Vectors

3D Review

Questions

- What is the distance formula for the distance between two points in space, (x_0, y_0, z_0) and (x_1, y_1, z_1)?
- Find the distance between $(1, 2, 3)$ and $(-4, 4, 2)$.

Questions

Consider the set of points (x, y, z) that are a distance of 2 from the origin $(0, 0, 0)$.

- What is an equation that x, y, z must satisfy for (x, y, z) to be in this set?
- What is this shape?
- Generalize this to find an equation for any of this shape.

Questions

- The equation $(x-2)^2 + y^2 + (z+3)^2 = 4$ represents a sphere.
  - What is its center?
  - What is its radius?
  - Describe its intersections with each of the coordinate planes.

Questions

Identify the set.

- $y = -2$
- The equation $x^2 + y^2 + z^2 + 2x - 4y - 10z = 0$ represents a sphere.
  - What is its center?
  - What is its radius?
- $z > 4$

Vectors

Definition

A vector is an object with direction and magnitude.

Example

- While driving at one particular instance my speedometer shows that I'm going 53 mph and my gps shows I'm headed 30° north of east. My ***velocity vector*** has magnitude (a ***speed*** -- 53 mph) and **direction** (30° north of east). This is a 2D vector.
- I'm standing at a particular point on a mountaintop. I need to tell a helicopter pilot how to reach me. I give her a direction and a distance to fly to reach me. The *position vector* from the helicopter to me has magnitude (the distance to fly to me) and the direction. This is a 3D vector.
- I need to move my refrigerator. To overcome friction I need to pull with a magnitude of 30 pounds and it needs to be away from the back wall in a perpendicular fashion. I need to pull with a force vector that has magnitude (30 pounds) and direction (directly away from the wall).

Vector graphics

Represent a vector with an arrow whose length is the magnitude of the vector and it points in the direction of the vector

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Graphics3D[{Arrow[{{0, 1, 2}, {2, 1, 3}}]}, Boxed → True, Axes -> True]
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Questions

Let (x_0, y_0) be the initial point of the arrow and (x_1, y_1) be the terminal point of the arrow where the arrowhead is.

- Draw a rectangle with the vector as one of the diagonals.
- What are the dimensions of the rectangle?
- What is the length of the arrow?
- Find the direction of the vector as an angle with the positive x-direction.

Vector components

The vector components for a vector are the dimensions of the rectangle if the vector is 2D, dimensions of the box if the vector is 3D.

Questions

- What are the components of the vector whose initial point (1, 2) and terminal point (3, 1)?

- What is the magnitude of this vector?
- Find the direction of this vector as an angle with the positive x-axis.

- What are the components of the vector whose initial point (4, 2, −3) and terminal point (1, 2, 3)?
  - What is the magnitude of this vector?

Notation

A vector is written in component form in two different ways. If vector v has x-component a, y-component b, and z-component c:

- $\vec{u} = \langle a, b, c \rangle$
- $\vec{u} = a\,\vec{i} + b\,\vec{j} + c\,\vec{k}$

The vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$ are called **unit vectors**, because they have unit length. They are also mutually perpendicular, and so they comprise what is called an **orthogonal basis** for 3-space. That's a mouthful. They're *orthogonal* because they're mutually perpendicular; they're a *basis* because one can write any vector in 3-space in terms of them.

Questions

- Draw vector $\langle 1, -2 \rangle$ with its initial point at (0, 0).
- Draw vector $2\,\vec{i} + 3\,\vec{j}$ with its initial point at (−1, 1).

The vectors $\vec{\mathrm{i}}$ and $\vec{\mathrm{j}}$ are an **orthogonal basis** for 2-space.

Vector arithmetic

Arithmetic is done on vectors only when it has geometric significance.

Scalar multiplication

A scalar is just a single number as opposed to a vector that is multiple information packaged in one object. Scalar multiplication:

scalar * vector = vector

Questions

- Let λ be the scalar 2 and $\vec{v} = \langle 1, 2 \rangle$. What should the components of $2\,\vec{v}$ be?
- Draw both $\vec{v}$ and $2\,\vec{v}$ with the same initial point. How are the two related geometrically?
- Draw both $\vec{v}$ and $-\frac{1}{2}\,\vec{v}$ with the same initial point. How are the two related geometrically?

Questions

- In general, how should $\lambda \langle a, b, c \rangle$ be computed?
- What is the geometric significance of $\lambda \vec{v}$ as compared to $\vec{v}$.
- Given the three vectors $\vec{v}_1 = \langle 4, 12, -8 \rangle$, $\vec{v}_2 = \langle -3, -9, 6 \rangle$, $\vec{v}_3 = \langle 2, 2, -6 \rangle$, are any parallel to each other?

Vector addition

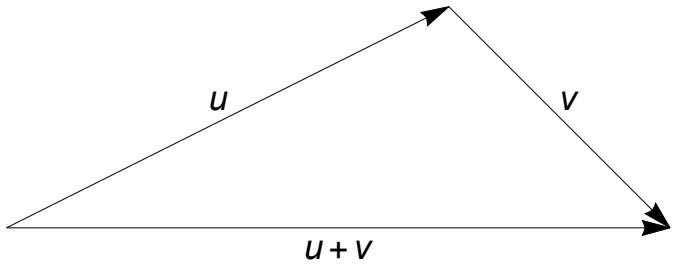
Component form

If $\vec{u} = \langle a, b, c \rangle$ and $\vec{v} = \langle e, f, g \rangle$, then $\vec{u} + \vec{v} = \langle a + e, b + f, c + g \rangle$

Geometric significance

Equivalent forms:

- Draw vector $\vec{u}$. Place the initial point of vector $\vec{v}$ at the terminal point of $\vec{u}$. Draw the vector sum with its initial point at the initial point of $\vec{u}$ and its terminal point at the terminal point of $\vec{v}$.



- Draw vectors $\vec{u}$ and $\vec{v}$ with initial points together. The two vectors form two sides of a parallelogram. Complete the parallelogram. Draw the vector sum from the initial points of $\vec{u}$ and $\vec{v}$ to the opposing vertex of the parallelogram.

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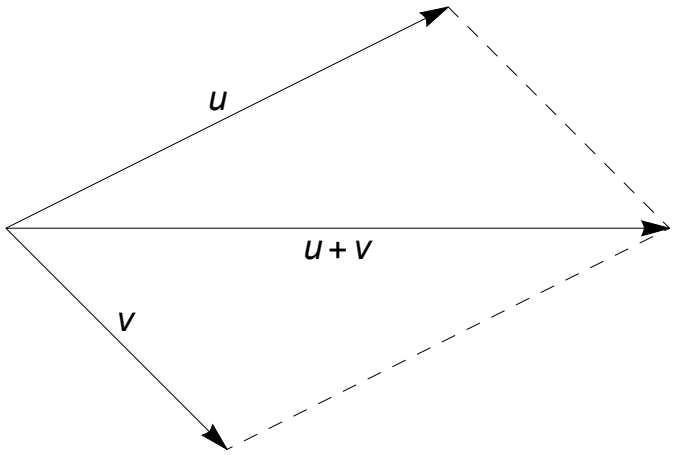

Questions

Let $\vec{u} = \langle 1, 1 \rangle$ and $\vec{v} = \langle 1, 2 \rangle$.

- Draw $\vec{u} + \vec{v}$ as in the first geometric significance.
- Draw $\vec{u} + \vec{v}$ as in the second geometric significance.

Questions

Using the same two vectors, what is the geometric significance of $\vec{u} - \vec{v}$? Draw it.

Questions

Consider the triangle with vertices $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$.

- What is vector $\vec{AB}$ in component form? What is vector $\vec{BC}$ in component form?
- What is the vector sum $\vec{AB} + \vec{BC}$ in component form?
- Geometrically what is the vector sum $\vec{AB} + \vec{BC}$?

Question

Why is it true that the line joining the midpoints of two sides of a triangle must be parallel to the third side?

Question

There are the two component notations for vectors, $\vec{u} = \langle a, b, c \rangle$ and $\vec{u} = a\,\vec{i} + b\,\vec{j} + c\,\vec{k}$. Expressions $\vec{i}$, $\vec{j}$, $\vec{k}$ are vectors themselves. What are they?

Unit vectors

Definition

A *unit vector* is a vector with magnitude 1. Its magnitude is fixed, but its direction can be any direction.

Questions

Let $\vec{u} = \langle 1, 2 \rangle$

- Is $\vec{u}$ a unit vector?
- Find a scalar λ such that $\lambda \vec{u}$ is a unit vector.
- Find a general method for taking a vector and finding a unit vector pointing in the same direction.
- Does this work for all vectors?

Questions

- What are the unit vectors parallel to the tangent line to the curve $y = 2\sin(x)$ at the point $(\pi/6, 1)$?
- What are the unit vectors perpendicular to the tangent line to the curve $y = 2\sin(x)$ at $x = \pi/6$?