Section 12.3: Dot Product

Vector arithmetic review

Questions

- Find a unit vector that points in the same direction as $\vec{v} = \langle 1, 2, -3 \rangle$.
- Find a vector of length 5 that points in the opposite direction of $\vec{w} = \langle 1, 0, -3 \rangle$.

Vector products

Question

If the world were fair how would you compute $\langle a, b, c \rangle \langle e, f, g \rangle$?

Other products

- The *dot product* is a product between two vectors. The product is a scalar, not a vector. This works for either 2D vectors or 3D vectors.
  $$\text{vector}_1 \cdot \text{vector}_2 = \text{scalar}$$
- The *cross product* is also a product between two vectors. The product is a vector, but it only works for 3D vectors.
  $$\text{vector}_1 \times \text{vector}_2 = \text{vector}$$

Dot product

Component definition

The dot product of $\vec{u} = \langle a, b, c \rangle$ and $\vec{v} = \langle d, e, f \rangle$ is

$$\vec{u} \cdot \vec{v} = \langle a, b, c \rangle \cdot \langle d, e, f \rangle = a\,d + b\,e + c\,f$$

Questions

- What is $\langle 1, 2, 3 \rangle \cdot \langle 1, -2, 1 \rangle$?
- What is $\langle 1, 4 \rangle \cdot \langle 5, -2 \rangle$?
- If $\vec{u} = \langle 2, 1, 4 \rangle$ what is $\vec{u} \cdot \vec{u}$?

Geometric definition

The dot product of $\vec{u}$ and $\vec{v}$ is defined in terms of the angle between the two vectors, call it θ.

$$\vec{u}\cdot\vec{v} = \left|\vec{u}\right| \left|\vec{v}\right| \cos(\theta)$$

Questions

Let $\vec{u} = \langle 2, 0\rangle$ and $\vec{v} = \langle 3, 3\rangle$.

- What is the angle between these two vectors?
- Compute $\vec{u}\cdot\vec{v}$ using the geometric definition.

Equivalence of definitions

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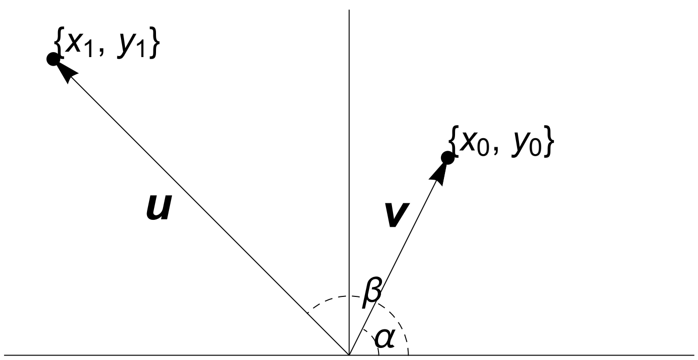


In the above

α is the angle between the positive x-axis and vector $\vec{v}$

(x_0, y_0) is the terminal point of vector $\vec{v}$

β is the angle between the positive x-axis and vector $\vec{u}$

(x_1, y_1) is the terminal point of vector $\vec{u}$

- What is the angle between vectors $\vec{u}$ and $\vec{v}$ in terms of the angles α and β?
- What are the polar coordinates of (x_0, y_0) and (x_1, y_1) in terms of the other variables in the plot?
- What is the trigonometric identity on the cosine of the difference of two angles?

Main uses of dot product

We compute the dot product using the component formulation, while applications arise from the geometric formulation. Besides determining the angle between two given vectors:

- What is the angle between a vector and itself? Use your answer to give geometric significance to $\vec{u}\cdot\vec{u}$.
- Given two vectors how can you use the dot product to determine if they are perpendicular to each other?

Questions

- What is the angle between ⟨1,2,3⟩ and ⟨1,-2,1⟩?
- Which of the following vectors are perpendicular to each other?
  - ⟨1, −1, 1⟩
  - ⟨1, 2, 1⟩
  - ⟨1, 1, 1⟩
  - ⟨−3, 0, 3⟩
- What choice(s) for k will make vector ⟨1, 2, k⟩ perpendicular to vector ⟨2, 1, −2⟩?

Projections

The idea of projections comes up in many different contexts.

Examples

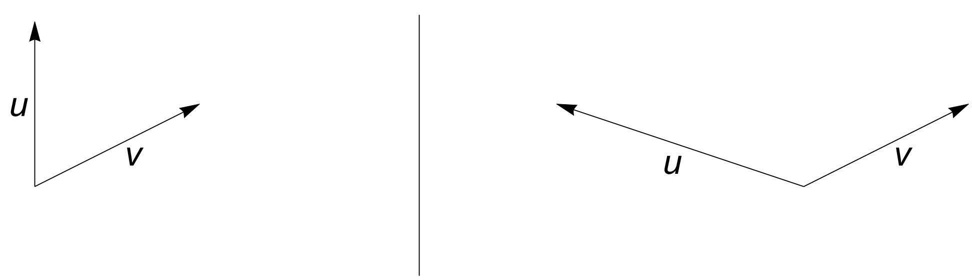
- In computer graphics depicting 3D objects on a 2D screen involves projecting.
- In mechanics the result of a force might be constrained by how the object feeling the force is allowed to move. The force is projected into the allowable motion.
- In data science a huge amount of data must be reduced to a more manageable amount of data. Typically, this involves projecting the structure of the more complicated data into a less complicated structure using some kind of projection.

These are all built from the idea of projecting one vector onto another.

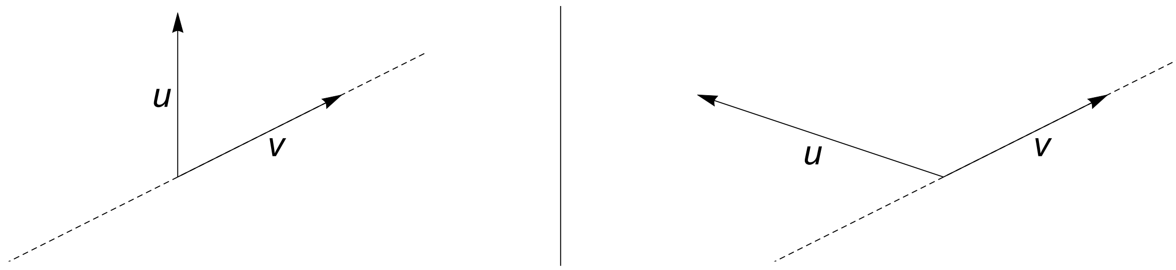
Vector projection

Given two vectors $\vec{u}$ and $\vec{v}$ the projection of $\vec{u}$ onto $\vec{v}$ is a new vector that has the same (or opposite) direction as $\vec{v}$ but whose length comes by finding the “perpendicular” shadow of $\vec{u}$ on $\vec{v}$.

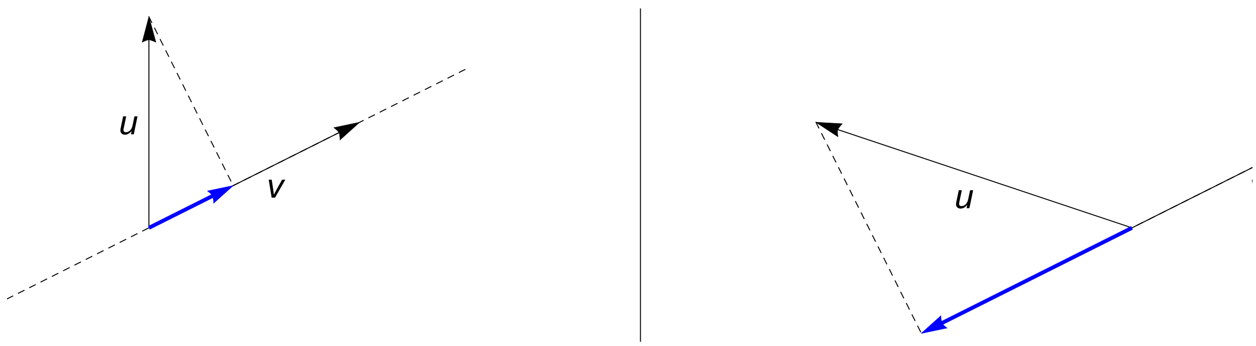
- Place the two vectors so they have the same initial point.



- Draw the line through the vector you are projecting onto, here vector $\vec{v}$.



- Draw the line passing through the terminal point of vector $\vec{u}$ that is perpendicular to the line through $\vec{v}$. The projection is the vector whose initial point is the same as the original two vectors and whose terminal point is the intersection of the perpendicular line with the line through $\vec{v}$.



Questions

- If θ is the angle between $\vec{u}$ and $\vec{v}$, what is the length of the projected vector in terms of θ?
- Scale vector $\vec{v}$ by the appropriate amount to get the projected vector.

Questions

Let $\vec{u} = \langle 1, 2, -2 \rangle$ and $\vec{v} = \langle 0, 1, 3 \rangle$.

- What is $\text{proj}_{\vec{v}}\,\vec{u}$?
- What is $\text{proj}_{\vec{u}}\,\vec{v}$?

Work

If a constant force vector $\vec{F}$ is exerted in a straight line from point P to point Q, the work done by the force is

$$W = \vec{F} \cdot \vec{PQ}$$

Question

1. A wagon is pulled a distance of 100 meters along a horizontal path by a constant force of 70 Newtons. The handle of the wagon is held at an angle of 35° above the horizontal. What is the work done by that force?

Parametrics with vectors, and dot products

1. We want to design a clock, using vector “hands” that tick off the seconds (sweep), minutes (big hand), and hours (little hand). What equations would you use for the vector hands? Measure time in minutes, and all hands start at the top of the clock (along the y-axis) at midnight.
2. When will the hands of the hour and minute hands be parallel?