Section 12.4: Cross Product

Review

Questions

- What are the components of the vector that points from point $(1, 0, -2)$ to point $(2, 4, 3)$?
- What is a unit vector that points in that same direction?

Questions

- What is the geometric significance of scalar multiplication?
- What is the geometric significance of vector addition?
- What is the geometric significance of vector subtraction?
- What is the geometric significance of the dot product?

Cross product definition

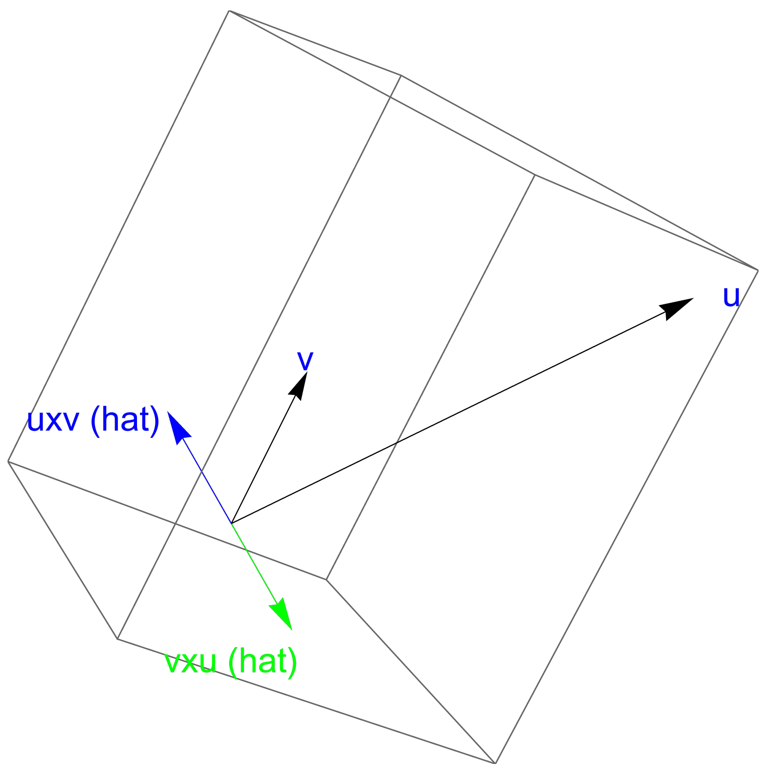
The dot product is a multiplication-like operation between two vectors (two 2D vectors or two 3D vectors) that gives a scalar value.

The *cross product* is a multiplication-like operation between two 3D that gives a vector value.

Geometric definition

The cross product of 3D vectors $\vec{u}$ and $\vec{v}$ is denoted $\vec{u} \times \vec{v}$.

- The magnitude of $\vec{u} \times \vec{v}$ is given by
$$\left| \vec{u} \times \vec{v} \right| = \left| \vec{u} \right| \left| \vec{v} \right| \sin(\theta)$$
where θ is the smaller of the two angles between vectors $\vec{u}$ and $\vec{v}$.
- The direction of $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. There are two such directions. Choose the one that satisfies the right-hand rule.



Right-hand rule

To determine the direction of $\vec{u} \times \vec{v}$ from the two possible directions, using the fingers on your right hand

- Point your index finger in the direction of $\vec{u}$.
- Sweep that finger towards the direction of $\vec{v}$. Your thumb will point in the correct direction of $\vec{u} \times \vec{v}$.

Questions

Let $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, and $\vec{k} = \langle 0, 0, 1 \rangle$.

- What is $\vec{i} \times \vec{j}$?
- What is $\vec{i} \times \vec{k}$?
- What is $\vec{j} \times \vec{k}$?

Question

Are $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ the same?

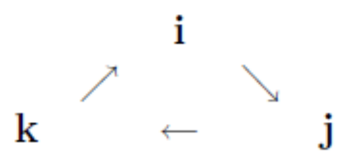
Component definition

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

So it's products of the components of the vectors from the other dimensions (e.g. y,z) that figure in the component for x, say (indices 2 and 3 are "yz" indices, and so on).

And if we check the **permutations** of the indices, then 123, 231, and 312 give rise to positive contributions, while 321, 213, and 132 give rise to negative contributions.



Questions

Let $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, and $\vec{k} = \langle 0, 0, 1 \rangle$.

- What is $\vec{i} \times \vec{j}$?
- What is $\vec{i} \times \vec{k}$?
- What is $\vec{j} \times \vec{k}$?

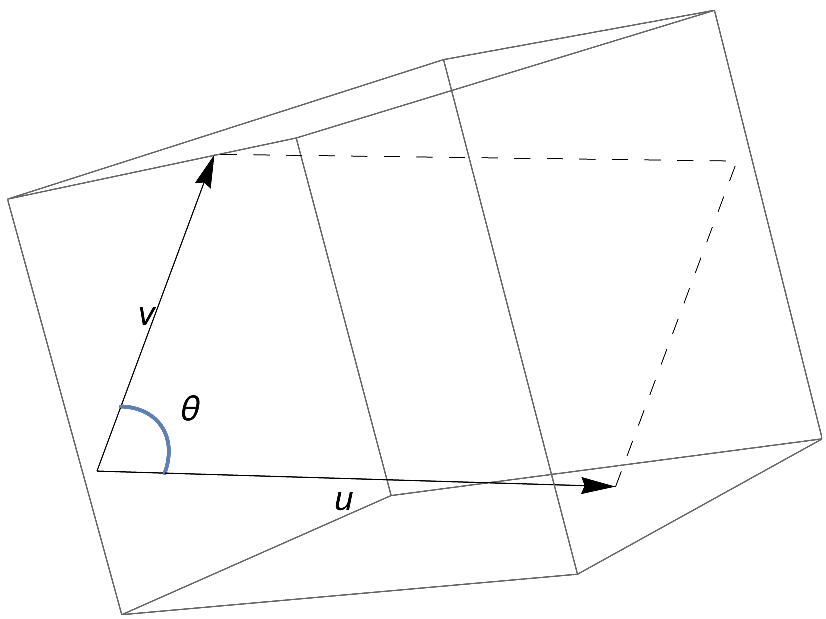
Question

What is a vector that is perpendicular to both $\vec{u} = \langle 2, -1, 3 \rangle$ and $\vec{v} = \langle 1, 2, 1 \rangle$?

Cross product geometric properties

Questions

- What must be true about vectors $\vec{u}$ and $\vec{v}$ if $\vec{u} \times \vec{v} = 0$?
- Place vectors $\vec{u}$ and $\vec{v}$ so there initial points are the same. They form two of the sides of a parallelogram. What is the area of this parallelogram in terms of the lengths of these vectors and the angle between them?



- What is the area of the **parallelogram** formed from vectors $\vec{u} = \langle 2, -1, 3\rangle$ and $\vec{v} = \langle 1, 2, 1\rangle$?
- What is the area of the **triangle** with vertices (0, 1, 0), (2, −1, −1), (−1, 0, 1)?

Cross product algebraic properties

Question

How are $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ related? (The cross-product is non-**commutative.**)

Other properties

1. $(s\,\vec{a}) \times \vec{b} = s(\vec{a} \times \vec{b}) = \vec{a} \times (s\,\vec{b})$ (where s is a scalar)

2. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

3. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

Questions

- Show property 2 is true by computing each side using
  $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$, $\vec{c} = \langle c_1, c_2, c_3\rangle$
- Show that the cross-product is non-**associative**: that is, that
  $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$
  (in general).

- What is $\vec{k} \times \left(\vec{i} + 2\,\vec{j}\right)$?
- What is $\vec{a} \times \vec{b} = \left(a_1\,\vec{i} + a_2\,\vec{j} + a_3\,\vec{k}\right) \times \left(b_1\,\vec{i} + b_2\,\vec{j} + b_3\,\vec{k}\right)$?