Section 6.8: Indeterminate Forms

Review

Trigonometric functions:

Questions

- What is the domain and range of $\sin^{-1}(x)$? Of $\tan^{-1}(x)$? Of $\cos^{-1}(x)$?
- What is $\tan(\tan^{-1}(0.3))$? What is $\cos^{-1}(\cos(\pi/5))$? What is $\sin(\cos^{-1}(1/2))$?

Right triangles

To find $\tan(\sin^{-1}(0.4))$, let $\theta = \sin^{-1}(0.4)$ so that $0.4 = \sin(\theta)$. Represent $\sin(\theta)$ in a right triangle.

Using this triangle, $\tan(\sin^{-1}(0.4)) = \tan(\theta) = \frac{0.4}{\sqrt{1-0.4^2}} = \frac{0.4}{\sqrt{0.84}} \approx 0.436$.

Questions

Use a right triangle to find a formula for $\sec(\tan^{-1}(x))$.

Indeterminate forms

Questions

- What is $\lim_{x\to 0} \frac{2x+1}{x-1}$?
- What is $\lim_{x\to 0} \frac{4x+1}{x}$?
- What is $\lim_{x\to 0} \frac{2x}{x}$?
- What is $\lim_{x\to 0} \frac{x}{5x}$?
- What is $\lim_{x\to \infty} \frac{x}{2}$?
- What is $\lim_{x\to \infty} \frac{1}{x}$?
- What is $\lim_{x\to -\infty} \frac{x-1}{x+1}$?
- What is $\lim_{x\to \infty} \frac{x^2}{x}$?

Given a limit $\lim_{x\to a} f(x)$, if we can simply evaluate $f(a)$ as the limit we say $\lim_{x\to a} f(x)$ is *determinate (*and *continuous).* If we cannot simply evaluate $f(x)$ at $x = a$, we say the limit is *indeterminate* -- it may or may not exist. Perhaps some algebra will help....

Indeterminate limits

- $\frac{0}{0}$, a small number divided by a small number could be anything. More work is needed.
- $\frac{\infty}{\infty}$, a large number divided by a large number could be anything. More work is needed.

L'Hopital's rule

If you have a limit of a quotient which is either a $\frac{0}{0}$ or an $\frac{\infty}{\infty}$ limit, then the following is true if the limit (and the derivatives) exists:

$$\lim_{x\to a} \frac{g(x)}{h(x)} = \lim_{x\to a} \frac{g'(x)}{h'(x)}$$

Warning: If the limit is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the above two limits are not equal.

Example

To evaluate $\lim_{x\to\infty} \frac{x}{e^x}$, first notice that plugging in ∞ for x produces $\frac{\infty}{e^\infty} = \frac{\infty}{\infty}$. We can use L'Hopital's rule:

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{(x)'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Questions

Evaluate the following limits.

- $\lim_{x\to\infty} \frac{x}{\ln(x)}$
- $\lim_{x\to\infty} \frac{\tan^{-1}(x)}{x^2+1}$
- $\lim_{x\to 1} \frac{x^2-3x+2}{\sin(x-1)}$
- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

Why it works

For the $\frac{0}{0}$ case, this means $f(a) = 0$ and $g(a) = 0$. Remember the limit definition of the derivative

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$$

Since $f(a) = 0 = g(a)$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{f'(a)}{g'(a)}$$

Questions

Can we use L'Hopital's rule on $\lim_{x\to 1}\frac{x-1}{e^{x-1}}$? Compare the actual value of this limit with the limit that comes from L'Hopital's rule.

Other indeterminate forms

There are other indeterminate forms. Sometimes we can use L'Hopital's rule to evaluate them *if* we can rewrite into either the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

Other forms

- $\infty - \infty$ or a large number minus a large number.
- $0\cdot\infty$ or a number close to zero times a large number.
- Indeterminate powers
  - 0^0 or a small number raised to another small number.
  - ∞^0 or a large number raised to a small number.
  - 1^∞ or a number close to 1 raised to a large power.

Difference example

Evaluate $\lim_{x\to\infty}(\ln(2x+1)-\ln(3x-5))$.

Use log properties to rewrite as a fraction.

$$\begin{aligned}&\lim_{x\to\infty}(\ln(2x+1)-\ln(3x-5))\\ &\quad=\lim_{x\to\infty}\ln\left(\frac{2x+1}{3x-5}\right)\\ &\quad=\ln\left(\lim_{x\to\infty}\frac{2x+1}{3x-5}\right)\ \text{-- (What allows us to do this?)}\\ &\quad=?\end{aligned}$$

Product example

Evaluate $\lim_{x\to\infty}x(\pi/2-\tan^{-1}(x))$.

Rewrite one of the factors as a fraction, factor $=\frac{1}{1/\text{factor}}$.

$$\lim_{x\to\infty}x(\pi/2-\tan^{-1}(x))=\lim_{x\to\infty}\frac{(\pi/2-\tan^{-1}(x))}{1/x}=?$$

Questions

- $\lim_{x\to\infty}x\sin\left(\frac{2}{x}\right)$
- $\lim_{x\to 0^+}x\ln(x)$

Power example

A limit that comes from finance is

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x$$

Change the exponential expression to base e and take the limit of the exponent.

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{r}{x}\right)^x\right)} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{r}{x}\right)}$$
$$= e^{\lim_{x\to\infty} x\ln(1+r/x)} = ?$$

Questions

Consider the function $g(x) = (x + e^x)^{1/x}$. It is defined on $(0, \infty)$.

- What is $\lim_{x\to 0^+} g(x)$?
- What is $\lim_{x\to\infty} g(x)$?