Section 7.4: Partial Fraction Decomposition

Rational functions

A **rational** function is a function that can be written as the **ratio** of two polynomials.

Questions

- Is $f(x) = \frac{x^3+1}{x^2}$ a rational function? What is $\int \frac{x^3+1}{x^2}\,dx$?
- Is $g(x) = x$ a rational function? What is $\int x\,dx$?
- Is $h(x) = \frac{1}{x}$ a rational function? What is $\int \frac{1}{x}\,dx$?
- Is $j(x) = \frac{1}{1+x^2}$ a rational function? What is $\int \frac{1}{1+x^2}\,dx$?
- Is $k(x) = \frac{x^3-3x^2+4x-3}{x^2-3x+2}$ a rational function? What is $\int \frac{x^3-3x^2+4x-3}{x^2-3x+2}\,dx$?

Question

- If you knew that $\frac{3x^5-2x^4-36x^3-25x^2-41x-49}{x^4-x^3-11x^2-x-12} = 3x+1-\frac{3}{x-4}+\frac{1}{x+3}+\frac{2}{1+x^2}$, could you evaluate $\int \frac{3x^5-2x^4-36x^3-25x^2-41x-49}{x^4-x^3-11x^2-x-12}\,dx$?

Partial fraction decomposition

This is a technique for rewriting any rational function in a form in which it can be integrated.

Step 1

If the degree of the numerator is greater than or equal to the degree of the denominator, use long division to rewrite the rational function.

Example

Using long division on the rational function $\frac{4x^3-x}{2x^2+1}$ we get

$$\begin{array}{r|l} & 2x \\ \hline 2x^2 + 1 & 4x^3 - x \\ & -(4x^3 + 2x) \\ \hline & 0 - 3x \end{array}$$

so that $\frac{4x^3-x}{2x^2+1} = 2x + \frac{-3x}{2x^2+1}$.

Question

- What does long division say that $\frac{x^2+6}{x^2-3x+2}$ is equal to?

Step 2

Factor the denominator. In theory, any polynomial can be factored into linear terms $m x + b$ and quadratic terms $a x^2 + b x + c$. (You'd rather have linear factors -- but it's not really up to you! It's for your polynomial to decide.)

Questions

What is the factorizations for the denominators of each of the following rational functions?

- $\frac{1}{x^3+5x^2+4x}$
- $\frac{2x-1}{x^3+4x}$
- $\frac{8x^2+x-4}{5x^3-2x^2}$

Step 3

- If the denominator is factored into all different linear terms, rewrite as follows:
$$\frac{p(x)}{(a_1 x-b_1)(a_2 x-b_2)\ldots(a_n x-b_n)} = \frac{A_1}{a_1 x-b_1} + \frac{A_2}{a_2 x-b_2} + \ldots + \frac{A_n}{a_n x-b_n}$$
- If the denominator has some quadratic factors that couldn't be factored further, rewrite as follows:
$$\frac{p(x)}{(a x^2+b x+c)\, q(x)} = \frac{A x+B}{a x^2+b x+c} + \text{rest decomposed}$$
- If the denominator has a linear term to a power, rewrite as follows:
$$\frac{p(x)}{(a x+b)^n q(x)} = \frac{A_1}{(a x+b)^n} + \frac{A_2}{(a x+b)^{n-1}} + \ldots + \frac{A_{n-1}}{(a x+b)^2} + \frac{A_n}{a x+b} + \text{rest decomposed}$$

Example

The rational function $\frac{x}{(2x-1)(x+5)^2(x^2+x+1)(3x^2+8)}$ can be rewritten as
$$\frac{x}{(2x-1)(x+5)^2(x^2+x+1)(3x^2+8)} = \frac{A}{2x-1} + \frac{B}{x+5} + \frac{C}{(x+5)^2} + \frac{D x+E}{x^2+x+1} + \frac{F x+G}{3x^2+8}$$
for some numbers A, B, C, D, E, F, G.

Questions

What would be the decomposed form of the following rational function? (Using letters in the numerators -- don't solve for them yet!)

- $\frac{3x^2+7x-8}{(2x+5)(7x-1)(x+10)}$
- $\frac{x^2+6x+13}{(x-5)^3(x+1)(x^2+10)}$

Step 4

Clear the denominators by cross multiplying. Then either

a. Plug in specific values of x into the ensuing equations or their derivatives to obtain equations on the unknown numbers. Solve to determine those numbers.

b. match the coefficients of different powers of x of the numerators. Since the numerators on both sides must be the same for all values of x (since the denominators are the same) , the coefficients of different powers of x must be the same. You can see this by thinking about matching their derivatives (first, second, third, etc.).

Example

$\frac{3x+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

$\longrightarrow 3x+1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

- Evaluate this equation for $x=0$ to get $1 = -A + 0 + 0$ or $A = -1$.
- Evaluate this equation for $x=1$ to get $4 = 0 + 2B + 0$ or $B = 2$.
- Evaluate this equation for $x=-1$ to get $-2 = 0 + 0 - 2C$ or $C = -1$.

This means $\frac{3x+1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{2}{x-1} + \frac{-1}{x+1}$

Question

What is the value of $\int \frac{3x+1}{x(x-1)(x+1)} dx$?

Question

Using this technique, determine what the decomposition of $\frac{5x-7}{x^2(x+4)}$ is.

The easy way to do this is to match the coefficients of different powers of x. Since the numerators must be the same for all values of x, the coefficients of different powers of x must be the same.

Question

Using this technique, determine what the decomposition of $\frac{x^2+x+1}{(x-1)(x^2+4)}$ is.

Questions

Using partial fraction decomposition, evaluate the following integrals. (Don't forget step 1.)

- $\int \frac{x^3+x}{x-1}\,dx$
- $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$
- $\int \frac{x^4-2x^2+4x+1}{(x-1)^2(x+1)}\,dx$
- $\int \frac{2x^2-x+4}{x^3+4x}\,dx$