Section 7.8: Improper Integration

Review

To evaluate a definite integral for which we can't find a "nice" antiderivative use numerical integration.
Let $x_k = a + k\left(\frac{b-a}{n}\right)$ for $k = 0, 1, 2, \ldots, n$.

- LRR rule:

  $$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{k=1}^{n} f(x_{k-1})$$

- RRR rule (notice the subtle difference):

  $$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{k=1}^{n} f(x_k)$$

- Midpoint rule:

  $$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right),$$

  Error $\leq \frac{K(b-a)^3}{24\,n^2}$

- Trapezoid rule: is really just the average of the LRR and RRR rules, which are easily calculated. Nonetheless, one formula for Trapezoidal is $\int_a^b f(x)\,dx \approx \frac{b-a}{2\,n}\left(f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right)$,

  Error $\leq \frac{K(b-a)^3}{12\,n^2}$

- Simpson's rule: $\int_a^b f(x)\,dx$,

  $$\approx \frac{b-a}{3\,n}\left(f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + \ldots + 4\,f(x_{n-1}) + f(x_n)\right)$$

  $$= \frac{b-a}{3\,n}\left(f(x_0) + 4\sum_{k=1}^{n/2} f(x_{2\,k-1}) + 2\sum_{k=1}^{n/2} f(x_{2\,k})\right)$$

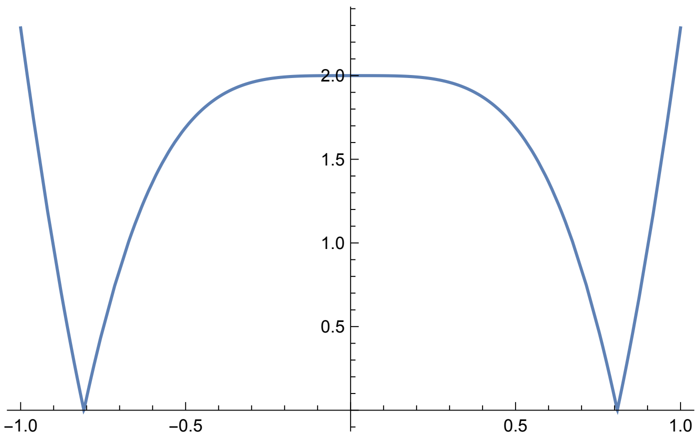
  Error $\leq \frac{L(b-a)^5}{180\,n^4}$

Here, K is an upper bound for $|\,f''(x)\,|$ for $a \leq x \leq b$, and L is an upper bound for $\left|\,f^{(4)}(x)\,\right|$ for $a \leq x \leq b$.
Again, Simpson's rule is better thought of as an average of Midpoint and Trapezoidal, but a **weighted** average: $S_{2\,n} = \frac{2\,M_n + T_n}{3}$

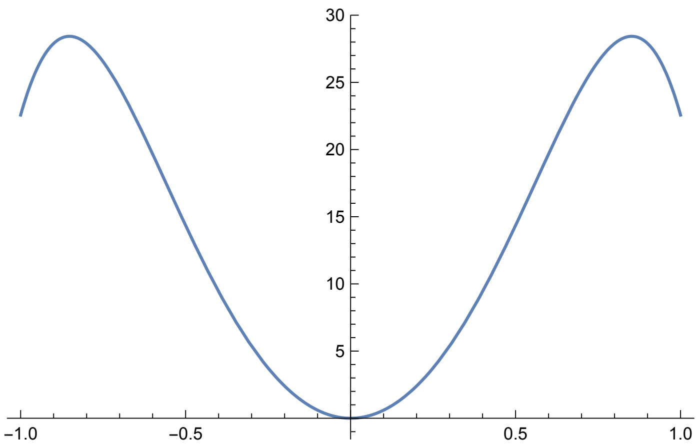
Questions

Consider the definite integral $\int_{-1}^{1} \sin(x^2)\,dx$. Let $f(x) = \sin(x^2)$.

- A plot of $|\,f''(x)\,|$ for $-1 \leq x \leq 1$ is shown below. Use it to determine a value of n for which the trapezoid rule will give an approximation to this integral with error less than 0.001.



- A plot of $\left| f^{(4)}(x) \right|$ for $-1 \le x \le 1$ is shown below. Use it to determine a value of n for which Simpson's rule will give an approximation to this integral with error less than 0.001.



Improper integrals

An improper integral is a definite integral where either one or both of the limits is ∞, or the integrand is not defined for some value(s) of x between the limits.

- Proper integral: $\int_1^{10} \frac{1}{x^2}\, dx$
- Improper integral: $\int_1^{\infty} \frac{1}{x^2}\, dx$
- Improper integral: $\int_0^{10} \frac{1}{x^2}\, dx$

Make sense out of an improper integral by turning it into a limit problem.

- Improper integral: $\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{R \to \infty} \int_1^{R} \frac{1}{x^2}\, dx$
- Improper integral: $\int_0^{10} \frac{1}{x^2}\, dx = \lim_{r \to 0^+} \int_r^{10} \frac{1}{x^2}\, dx$

Questions

- What makes $\int_0^{\infty} \frac{1}{1+x^2}\, dx$ an improper integral? Does it converge or not?

- What makes $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ an improper integral? Does it converge or not?
- What makes $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$ an improper integral? Does it converge or not?

Questions

Let S be the region between the x-axis and $y = \frac{1}{x}$ for $x \geq 1$.

- Is the area of S finite or not?
- Rotate S about the x-axis to create a solid of revolution. Is the volume of this solid finite or not?

Questions

- Evaluate $\int_1^\infty \frac{1}{x^3}\,dx$, $\int_1^\infty \frac{1}{x}\,dx$, and $\int_1^\infty \frac{1}{x^{1/3}}\,dx$.
- Evaluate $\int_0^1 \frac{1}{x^3}\,dx$, $\int_0^1 \frac{1}{x}\,dx$, and $\int_0^1 \frac{1}{x^{1/3}}\,dx$.
- Evaluate $\int_0^1 ln(x)\,dx$, $\int_0^\infty e^{-x}\,dx$.

Here’s a manipulate command, to illustrate the consequence of changing the power on x:

```
Manipulate[Plot[1/x^n, {x, 0.1, 2}, PlotRange → {0, 2}, Filling → Axis,
  FillingStyle → RGBColor[1, 1, 0], PlotLabel → n], {n, .1, 4}]
```