Directions: Weights for problems are not equal. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it).

Don't erase (cross out, instead); that way, in case you've done something good, I can give you credit. I have scratch paper up front.

Good luck!

Problem 1. Consider $f(x) = 2^{(3x-1)^2}$.

$(3x-1)(3x-1)$

$9x^2 - 3x - 3x + 1$

$9x^2 - 6x + 1$

a. (6 pts) Solve the equation $f(x) = 16$ for x.

$16 = 2^{(3x-1)^2}$

$16 = 2^{(9x^2 - 6x + 1)}$ Don't expand - factoring takes work!

$2 \cdot 2 \cdot 2 \cdot 2$

$4 \quad 4$

$2^4 = 2^{9x^2-6x+1}$

$\sqrt{4} = \sqrt{(3x-1)^2}$

$3x - \cancel{1} = \pm 2$

$+1 \quad +1$

$\dfrac{3x}{3} = \dfrac{\pm 2 + 1}{3}$

$x = \dfrac{3}{3} \quad x = \dfrac{-1}{3}$

$\boxed{x = 1, -1/3}$

$2^{(3-1)^2} \quad 2^{(-1-1)^2}$

$2^{2^2} \quad 2^4 = 2^4$

b. (4 pts) Is the function f invertible? Why or why not?

No because it is not one to one since there is a quadratic involved. For instance, the inputs 1 and -1/3 both have the same output (16). Thus it does not pass the horizontal line test either.

good

Great.

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Problem 2. Let $f(x) = x\sin^{-1}(x)$, and let me remind you that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

a. (8 pts) Find an equation of the tangent line to the graph of f when $x = \frac{1}{2}$.

$\left(\frac{1}{2}, \frac{\pi}{12}\right)$ $\quad f'(x) = \sin^{-1}(x) + \frac{x}{\sqrt{1-x^2}}$ ✓

$$f'\left(\tfrac{1}{2}\right) = \frac{\pi}{6} + \frac{1/2}{\sqrt{1-(1/2)^2}} = \frac{\pi}{6} + \frac{1/2}{\sqrt{3/4}}$$

$$= \frac{\pi}{6} + \frac{1/2}{\sqrt{3}/2}$$

what is that?

$$y = \frac{\pi + 2\sqrt{3}}{6}\left(x - \frac{1}{2}\right) + \frac{\pi}{12}$$

$$= \frac{\pi}{6} + \frac{1}{\sqrt{3}} = \frac{\pi}{6} + \frac{\sqrt{3}}{3}$$

$$= \frac{\pi + 2\sqrt{3}}{6}$$

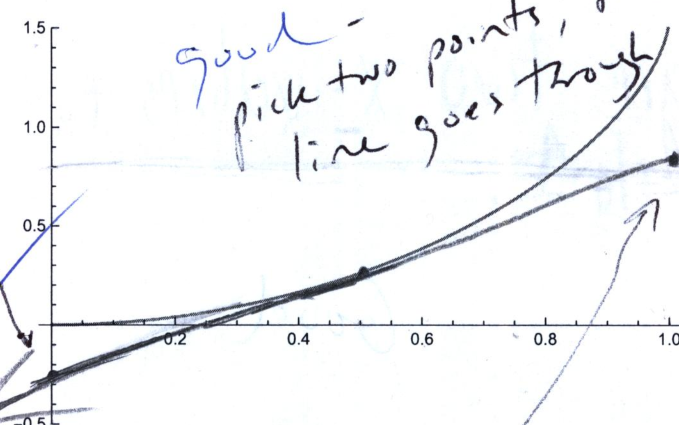
compute, to check that it seems reasonable.

b. (4 pts) Carefully graph your tangent line into the plot below:

$\frac{1}{2}, \frac{\pi}{12}$

$(0, -.289)$

$(1, .642)$



good - pick two points, & see if the line goes through the point of tangency!

Well done.

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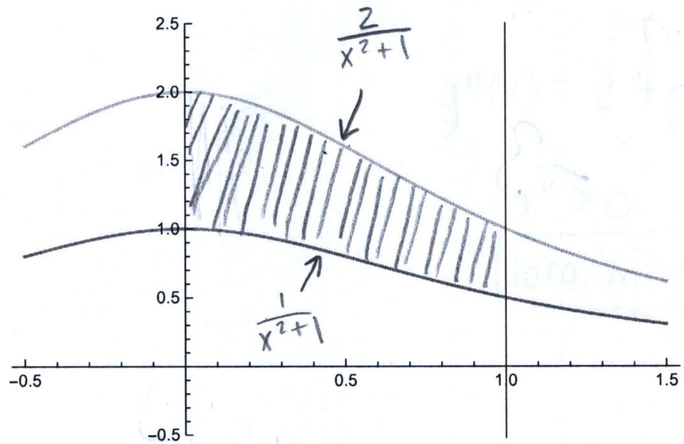
Problem 3. (10 pts) Find the area of the region bounded by the graphs of $y = \dfrac{1}{x^2+1}$ and $y = \dfrac{2}{x^2+1}$, and the lines $x = 1$ and the y-axis.



$$\int_0^1 \left(\frac{2}{x^2+1} - \frac{1}{x^2+1} \right) dx$$

$$\int_0^1 \left(\frac{1}{x^2+1} \right) dx$$

$$\tan^{-1}(x) \Big|_0^1$$

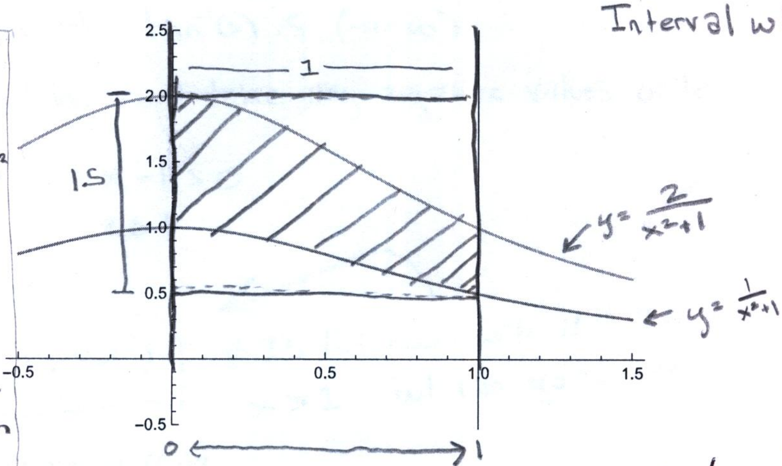
$$\tan^{-1}(1) - \tan^{-1}(0) = \boxed{0.79}$$

$= \pi/4$, to be precise.

Problem 3. (10 pts) Find the area of the region bounded by the graphs of $y = \dfrac{1}{x^2+1}$ and $y = \dfrac{2}{x^2+1}$, and the lines $x = 1$ and the y-axis.

Interval will be from [0, 1]

* The Area of the box I drew on the graph IS equal to $A_b = 1.5 \text{ unit}^2$ Since the Area region between the 2 equation only take up half-ish the Space of block my predictive Area of the Shaded region is around 0.75.



$f(x) = 2g(x)$

$2 \cdot g(x) - g(x)$

$= g(x)$!

Great!

$$A = \int_a^b (f(x) - g(x))\,dx$$

Top - bottom

Use algebra!

$$A = \int_0^1 \left[\left(\frac{2}{x^2+1}\right) - \left(\frac{1}{x^2+1}\right)\right] dx = 2\int_0^1 \left(\frac{1}{x^2+1}\right) dx - \int_0^1 \left(\frac{1}{x^2+1}\right) dx$$

$\tan^{-1}(1) = \theta$

$\Longleftrightarrow$

$\tan(\theta) = 1$

$\tan\left(\frac{\pi}{4}\right) = 1$

$$A = 2\tan^{-1}(x) - \tan^{-1}(x)\Big|_0^1$$

$$A = \left[2(\tan^{-1}(1)) - (\tan^{-1}(1))\right] - \left[2\tan^{-1}(0) - \tan^{-1}(0)\right]$$

0 0

$$A = \left[2\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right] = \frac{\pi}{4} = 0.785$$

$$\boxed{A = 0.785}$$

* This is really close to the predicted Area of my originall assessment So I am Confident

Well done.

Problem 4. Let $f(x) = \ln\left(\tan^{-1}(x-1)\right)$.

a. (6 pts) What is the domain of $f(x)$?

$0 = \tan^{-1}(x-1)$

$\tan(0) = x-1$

$0 = x-1$

$x = 1$

$f(0) = \ln(0) \rightarrow$ DNE

when $x=1$, $f(1)$ DNE

so domain of $f(x)$

is $(1,\infty)$

b. (6 pts) What is the range of $f(x)$?

$(-\infty, .4516)$ $(-\infty, \ln(\frac{\pi}{2}))$

great

$\lim_{x \to 0^+} \ln(x) = -\infty$

$\lim_{x \to \frac{\pi}{2}} \ln(x) \approx .4516 = \ln(\frac{\pi}{2})$! :)

approx. exact

c. (6 pts) Find a formula for the inverse function $f^{-1}(x)$.

$$y = \ln(\tan^{-1}(x-1))$$

$$f^{-1}(x) \rightarrow \quad x = \ln(\tan^{-1}(y-1))$$

$$e^x = e^{\ln(\tan^{-1}(y-1))}$$

$$e^x = \tan^{-1}(y-1)$$

$$\tan(e^x) = y-1$$

$$\boxed{y = \tan(e^x)+1}$$

bc

$$f(f^{-1}(x)) = \ln(\tan^{-1}(\tan(e^x)+1-1))$$

$$= \ln(\tan^{-1}(\tan(e^x)))$$

$$= \ln(e^x) = x$$

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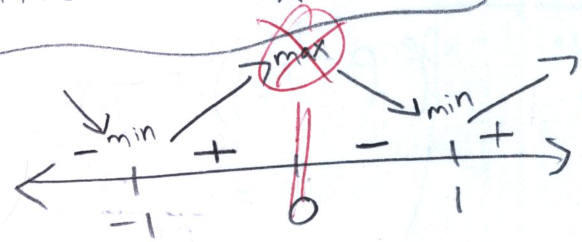
Problem 5. Let $g(x) = x^2 - \ln(x^2)$. $D_g = \mathbb{R} - \{0\}$

a. (8 pts) Find exact values for any critical numbers to $g(x)$. Determine whether each is an absolute or local extremum (max or min), or not. Give reasons for your conclusions.

$$g(x) = x^2 - \ln(x^2)$$
$$g'(x) = 2x - \frac{1}{x^2} \cdot 2x = 2x - \frac{2}{x} = 0$$
$$0 = 2x - \frac{2}{x} \rightarrow \frac{2}{x} = 2x$$
$$2 = 2x^2$$
$$1 = x^2$$

critical #'s $\rightarrow x = \pm 1$ + $g'(x)$ DNE @ $x = 0$

$$g'(x) = 2x - \frac{2}{x}$$



Nice work generally!

$\therefore$ both $x = 1$ + $x = -1$ are local minimums. Since $g(x)$ + $g'(x)$ DNE @ 0, $x = 0$ is an asymptote

Yes, but also absolute

Actually that doesn't necessarily follow – but true in this case

b. (6 pts) Find intervals of concavity for $g(x)$. Give reasons for your conclusions.

$$g(x) = x^2 - \ln(x^2)$$
$$g'(x) = 2x - 2x^{-1}$$
$$g''(x) = 2 + 2x^{-2} = 2 + \frac{2}{x^2} = 0$$
$$2 \neq \frac{2}{x^2}$$

$g''(x)$ is always positive from $(-\infty, \infty)$ + $\therefore g(x)$ is always concave up

Problem 5. Let $g(x) = x^2 - \ln\left(x^2\right)$.

a. (8 pts) Find exact values for any critical numbers to $g(x)$. Determine whether each is an absolute or local extremum (max or min), or not. Give reasons for your conclusions.

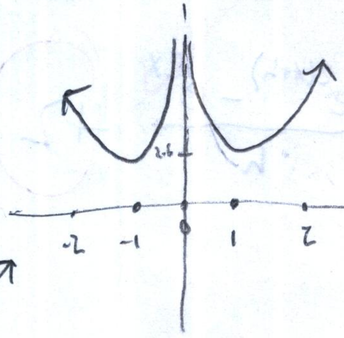
$g(x) = x^2 - 2\ln(x)$

$g'(x) = 2x - \frac{2}{x}$, set $g'(x) = 0$

$2x - \frac{2}{x} = 0 \to 2x = \frac{2}{x} \to 2x^2 = 2$

$x = \pm 1$

(changes sign @ $x = \pm 1$)



$\lim_{x \to 0} -\ln(x^2) = -(-\infty)$

$\lim_{x \to 0} -\ln(x^2) = \infty$

$\therefore$ $\boxed{\text{V.A. @ } x = 0}$

Well done

$g''(x) = 2 + \frac{2}{x^2}$ $\therefore$ $g''(x)$ is always positive

$\therefore$ $g(x)$ is always CC↑

$g(-1) = ?$

$g(1) = ?$

$g''(x) = 2 + \frac{2}{x^2}$, set $g''(x) = 0$

$2 + \frac{2}{x^2} = 0$

$\frac{2}{x^2} = -2$

$-2x^2 = 2 \to -x^2 = +1 \to x = \sqrt{-1} \to$ DNE

-.5 abs or local extreme?

	$-\infty, -1$	$-1, 0$	$0, 1$	$1, \infty$
$g'(x)$	$-$	$+$	$-$	$+$
$g''(x)$	$+$	$+$	$+$	$+$

$g'(-.5) = -1 + 4 = 3$

$g(0.5) = 1 - 4 = -3$

$g'(-2) = -4 + 1 = -3$

$g'(2) = 4 - 1 = 3$

Nice work!

b. (6 pts) Find intervals of concavity for $g(x)$. Give reasons for your conclusions.

$g''(x) = 2 + \frac{2}{x^2}$, $g''(x)$ cannot be negative (is never below x-axis) and therefore $g(x)$ is always concave up.

Problem 6. (10 pts) Use the limit definition of the derivative to compute the derivative $f'(x)$, where

$$f(x) = e^{2x}.$$

You may assume that $f'(0) = 2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{2x+2h} - e^{2x}}{h}$$

$$= \lim_{h \to 0} \frac{e^{2x}(e^{2h} - 1)}{h}$$

$$= e^{2x} \lim_{h \to 0} \frac{e^{2h} - 1}{h}$$

$$= e^{2x} \underbrace{\lim_{h \to 0} \frac{f(h) - f(0)}{h}}_{= f'(0)}$$

$$= f'(0) \cdot e^{2x} = \underline{2 \cdot e^{2x}}$$

Excellent

Problem 7. (4 points each) Find the limit, if it exists. Show your work.

a. $\lim_{x\to 0} \frac{x^2}{\cos(x)}$ $= \frac{0}{1} = \boxed{0}$

b. $\lim_{x\to -1} \frac{x+1}{\sin(x+1)}$ $= \frac{0}{0}$

$\lim_{x\to -1} \frac{1}{\cos(x+1)} = \frac{1}{1} = \boxed{1}$

c. $\lim_{x\to\infty} x\tan^{-1}\left(\frac{2}{x}\right)$ $= \infty \cdot 0$

$\lim_{x\to\infty} \frac{\tan^{-1}\left(\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\left(\frac{2}{x}\right)^2} \cdot -\frac{2}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-\frac{2}{\left(1+\left(\frac{2}{x}\right)^2\right)(x^2)}}{-\frac{1}{x^2}}$

$\lim_{x\to\infty} \cancel{x} \frac{2\cancel{x^2}}{\left(1+\frac{4}{x^2}\right)\cancel{x^2}} = \boxed{2}$

d. $\lim_{x\to\infty} \left(1+\frac{r}{x}\right)^x$

$\lim_{x\to\infty} e^{\ln\left(1+\frac{r}{x}\right)^x} = e^{\lim_{x\to\infty} x\ln\left(1+\frac{r}{x}\right)}$

$\lim_{x\to\infty} \frac{\ln\left(1+\frac{r}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{r}{x}}\left(\frac{r}{-x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{r\cancel{x^2}}{\left(1+\frac{r}{x}\right)\cancel{x^2}} = \frac{r}{1}$

$\boxed{e^r}$

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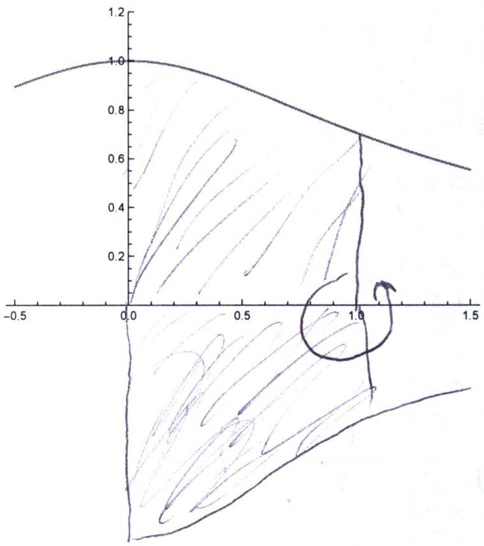
Problem 8. (10 pts) Find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = \dfrac{1}{\sqrt{x^2+1}}$ and the x-axis for $0 \le x \le 1$.



$y = \dfrac{1}{\sqrt{x^2+1}}$

$V = \pi \int_0^1 \left(\dfrac{1}{\sqrt{x^2+1}}\right)^2 dx$

$V = \pi \int_0^1 \dfrac{1}{x^2+1}\, dx$

$V = \pi \left(\tan^{-1}(x)\right)\Big|_0^1$

$V = \pi\left(\frac{\pi}{4}\right) - \pi(0) \rightarrow \boxed{V = \frac{\pi^2}{4} \text{ units}^3}$

exact!

$\boxed{V = 2.47 \text{ units}^3}$

approximate

$r \approx 1$

$h \approx 1$

$\pi = 3.14$

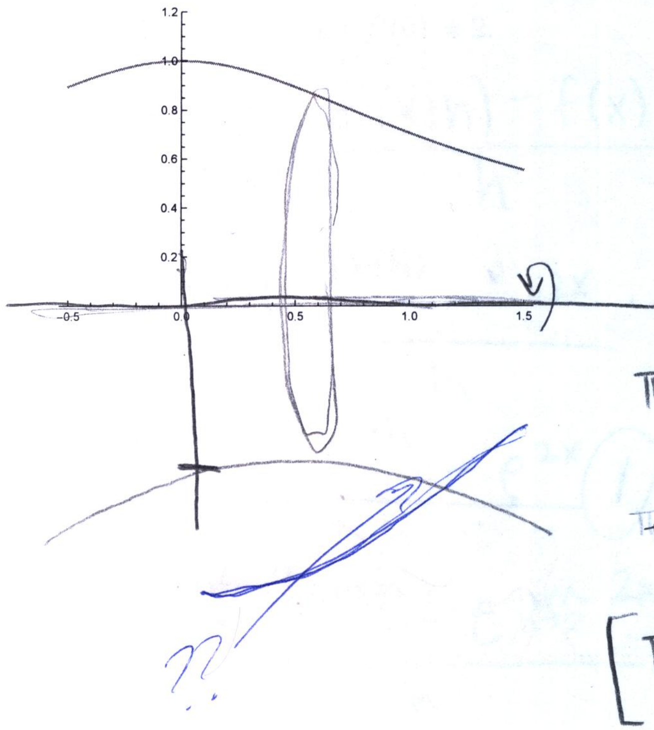
$V < 3.14$ as expected.

answer looks good.

Good!

Thank you!

Problem 8. (10 pts) Find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = \dfrac{1}{\sqrt{x^2+1}}$ and the x-axis for $0 \le x \le 1$.



$$\int_0^1 \pi \left(\frac{1}{\sqrt{x^2+1}}\right)^2 dx$$

$$\pi \int_0^1 \frac{1}{x^2+1}\, dx$$

$$\pi \int_0^1 \frac{1}{(x)^2+(1)^2}\, dx$$

$$\pi \cdot \frac{1}{1} \tan^{-1}\left(\frac{x}{1}\right)$$

$$\left[\pi \tan^{-1}(x)\right]_0^1$$

$$[\pi \tan^{-1}(1)] - [\pi \tan^{-1}(0)] = \boxed{2.4674011}$$

~~or π~~

$$2.4674011 - 0 =$$

Check my work

$V = \pi r^2 \cdot h$ radius $= 1.2$
height $= 1-0=1$

$\pi(1.2)^2 \cdot 1 = 4.52389$ ← bigger than the volume I calculated because radius is larger

Well done!

radius $= 0.5$
height $= 1$

$\pi(0.5)^2 \cdot 1 = 0.7853$ ← smaller than volume I calculated because radius is smaller

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