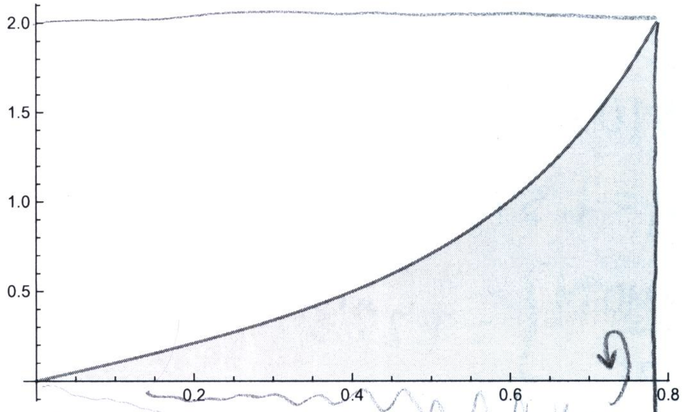
Problem 1 (10 pts): Compute the volume of the object obtained by rotating the function given by $f(x) = \tan(x)\sec^2(x)$ about the x-axis for $0 \le x \le \frac{\pi}{4}$.



~~Check~~ $\cancel{V = \pi(2.5)\left(\frac{\pi}{4}\right) = 6.1685}$

$(\tan(x)\sec^2(x))(\tan(x)\sec^2(x))$

$\pi \int_0^{\pi/4} \left[(\tan(x)\sec^2(x))^2\right] dx$

$\pi \int_0^{\pi/4} \left[\tan^2(x)\sec^4(x)\right] dx$

Well done.

$\pi \int_0^{\pi/4} \tan^2(x)\sec^2(x)\sec^2(x)\, dx$

$\pi \int_0^{\pi/4} \tan^2(x)(1+\tan^2(x))\sec^2(x)\, dx$

$u = \tan(x)$

$du = \sec^2(x)\, dx$

$\pi \int_0^{\pi/4} (\tan^2(x) + \tan^4(x))\sec^2(x)\, dx$

$\pi \int_0^1 (u^2 + u^4)\, du$

$u = \tan\left(\frac{\pi}{4}\right) = 1$

$\tan(0) = 0$

$\pi \left[\frac{1}{3}u^3 + \frac{1}{5}u^5\right]_0^1$

good!

Check

$\pi\left(\frac{1}{2}\right)^2\left(\frac{\pi}{4}\right) = 0.616$

* smaller ✓

$\pi\left(\left[\frac{1}{3}(1)^3 + \frac{1}{5}(1)^5\right] - \left[\frac{1}{3}(0)^3 + \frac{1}{5}(0)^5\right]\right)$

$\pi(2.5)^2\left(\frac{\pi}{4}\right) = 15.421$

* larger ✓

$\pi\left[\frac{8}{15} - 0\right] = \boxed{\frac{8\pi}{15} = 1.6755}$

I feel comfortable w/ my answer :)

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Problem 2 (25 pts): Consider the definite integral $I = \int_0^2 \frac{1}{\sqrt[3]{x^2+1}} dx$ The integrand doesn't have a "nice" antiderivative, so the integral can only be approximated numerically ($I \approx 1.5980$).

a. (12 pts) Your turn to approximate it: approximate this integral using left endpoint, right endpoint, midpoint and trapezoidal rules, with $n = 2$. Do the resulting errors make sense, given the figure?

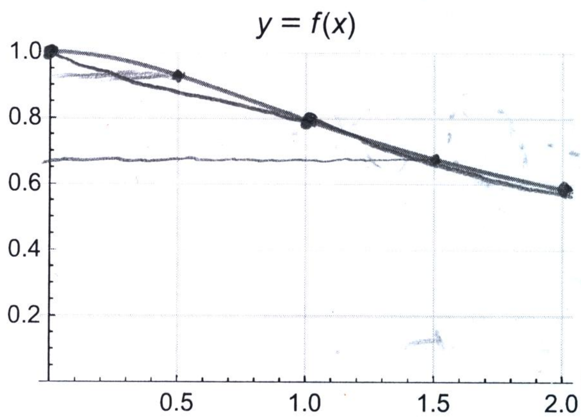
LRR = 1 + .8 (about) ≈ 1.7937

RRR = .8 + .6 (about about) ≈ 1.3785

Trap = $\frac{\text{LRR} + \text{RRR}}{2}$ = 1.5861

Mid = .9283 + .6751 = 1.6034

Nice work



method	estimate	error (estimate - 1.5980)
LRR	1.7937	0.1957
RRR	1.3785	-0.2195
trap	1.5861	-0.0119
mid	1.6034	0.0054

yes, they do make sense the Left end point would be an overestimate as the function is ~~decr~~ decreasing and that meants the right endpoint would be an underestimate trapezoid would also be an underestimate but more accurate with midpoint being even more accurate

b. (4 pts) Derive a Simpson's estimate from part a. What is its "n" (how many rectangles used)?

$S_4 = \frac{2\,\text{Mid} + \text{trap}}{3}$ (n=4) $\approx \boxed{1.5976}$

$\boxed{\text{error} = 0.0004}$

$\boxed{n=4}$

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1.377217
1.481248

a. (12 pts) Your turn to approximate it: approximate this integral using left endpoint, right endpoint, midpoint and trapezoidal rules, with $n = 2$. Do the resulting errors make sense, given the figure?

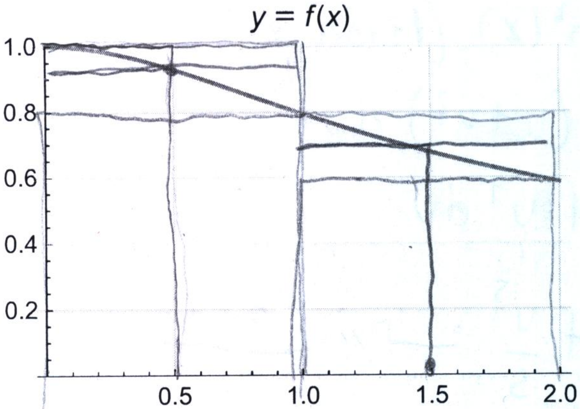
$n=2$

$LLR = f(0) + f(1.0) = 1.0 + .793 = 1.793$

$RRR = f(1.0) + f(2.0) = .793 + .584 = 1.377$

$Trap = \frac{1.793 + 1.377}{2} = 1.585$

$midpoint = f(.5) + f(1.5) = .9283 + .67511 = 1.60341$



yes for the most part they do. From the downward slope it is clear that left endpoint would be an overestimate and right would be an under. Also Trap is an under estimate ~~like usual~~ and mid is over estimate.

method	estimate	error (estimate - 1.5980)
LRR	1.793	.195
RRR	1.377	-.221
trap	1.585	-.013
mid	1.60341	.000545

b. (4 pts) Derive a Simpson's estimate from part a. What is its "n" (how many rectangles used)?

$\boxed{n=4}$

$$Simp_4 = \frac{2M + T}{3} = \frac{2(1.60341) + 1.585}{3} = 1.5972733$$

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c. **Problem 2**, cont. (9 pts) Using an appropriate error bound, what is the biggest error possible in midpoint, trapezoidal, and Simpson's estimates? Do your estimates and errors agree with your expectations? You may use the following graphs in your analysis:

$5\frac{1}{3}$

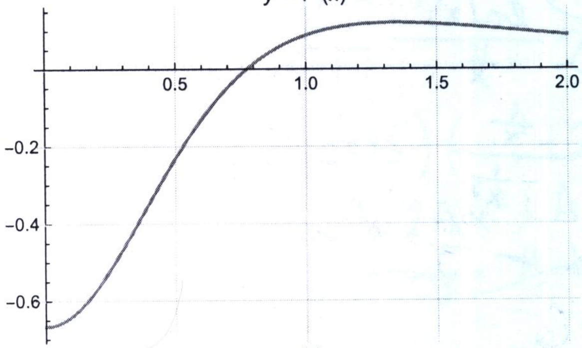
$f(x) = (x^2+1)^{-1/3}$

$f'(x) = -\frac{1}{3}(x^2+1)^{-4/3} \cdot 2x = -\frac{2}{3}x(x^2+1)^{-4/3}$

$f''(x) = \left(-\frac{2}{3}x\right)\left(-\frac{4}{3}(x^2+1)^{-7/3} \cdot 2x\right) + -\frac{2}{3}(x^2+1)^{-4/3}$

$f''(0) = -\frac{2}{3} \to \frac{2}{3}$

$y = f''(x) =$



$|E_T| \le \dfrac{\frac{2}{3}(2)^3}{12(2)^2}$ $\quad |E_T| \le \dfrac{\frac{16}{3}}{48}$

$|E_T| \le \frac{1}{9}$

$|E_M| \le \dfrac{\frac{2}{3}(2)^3}{24(2)^2}$ $\quad |E_M| \le \dfrac{\frac{16}{3}}{96}$

$|E_M| \le 0.0\overline{5}$

very good!

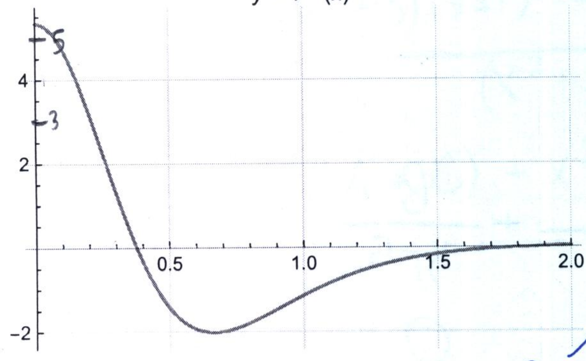
$f''(x) = \frac{16}{9}x^2(x^2+1)^{-7/3} - \frac{2}{3}(x^2+1)^{-4/3}$

$f'''(x) = \left(\frac{16}{9}x^2\right)\left(-\frac{7}{3} \cdot 2x(x^2+1)^{-10/3}\right) + (x^2+1)^{-7/3} \cdot \frac{32}{9}x$

$+ \frac{8}{9}(x^2+1)^{-7/3} \cdot 2x$

$-\frac{224}{27} = -8.\overline{296}\ x^3(x^2+1)^{-10/3} + \frac{32}{9}x(x^2+1)^{-7/3} + \frac{16}{9}(x^2+1)^{-7/3}$

$y = f^{(4)}(x)$



$f^4(x) = -\frac{224}{27}$

good guess!

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$|E_S| \le \dfrac{\frac{16}{3}(2)^3}{180(4)^2}$ $\quad |E_S| \le 0.0\overline{148}$

method	$\lvert error \rvert$ bound	actual $\lvert error \rvert$
trap	$0.\overline{1}$ or $1/9$	0.002
mid	$0.0\overline{5}$	0.005
simp	$0.0\overline{148}$	0.004

No ~~bc~~ the trapezoidal approx. should have had the smallest error bound due to my calculations but instead Simpson's rule did since it used more squares

Problem 3 (30 pts): (10 pts each) Apply appropriate techniques of integration to evaluate each integral:

a. $I = \int xe^x dx \to uv - \int v\,du$

$u = x \quad v = e^x$

$du = 1dx \quad dv = e^x dx$

$xe^x - \int e^x \cdot 1\,dx$

$\boxed{xe^x - e^x + C}$

c. **Problem 2**, cont. (9 pts) Using an appropriate error bound, what is the biggest error possible in midpoint, trapezoidal, and Simpson's estimates? Do your estimates and errors agree with your expectations? You may use the following graphs in your analysis:

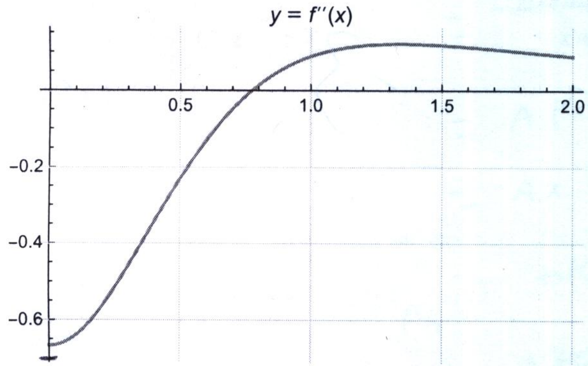
$K_2 = .7$

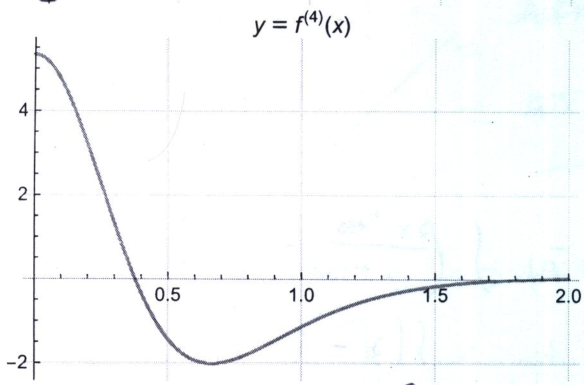
$$Err(M) = \frac{K(b-a)^3}{24N^2} = \frac{.7(2)^3}{24(2)^2} = \frac{5.6}{96} = .058$$

$$Err(T) = \frac{K(b-a)^3}{12N^2} = \frac{5.6}{12(2)^2} = .11667$$

$$Err(S) = \frac{K_4(b-a)^5}{180(N)^4} = \frac{5.5(2)^5}{180(2)^4} = \frac{176}{2880} = .061$$

$n=4$





$K_4 = 5.5$

method	|*error*| bound	actual |*error*|
trap	.11667	.002
mid	.058	.028
simp	.061	.018

~~Expected the~~ The estimates and errors agree with my expectations, although I did not expect the trapezoidal method to be the best, as it has the highest err. bound.

Due to the crudeness of your calculations (from the graph)

Problem 3 (30 pts): (10 pts each) Apply appropriate techniques of integration to evaluate each integral:

a. $I = \int xe^x dx$

$\int xe^x dx = xe^x - \int e^x dx$

$f(x)$, $g'(x)$

$f'(x) = 1$ $g(x) = e^x$

integrate by parts

$\int xe^x dx = xe^x - e^x = \boxed{e^x(x-1) + C}$

b. $J = \int_0^1 \ln(x)dx$

$f(x) = \ln(x)$ $\quad g'(x) = 1$

$f'(x) = \frac{1}{x}$ $\quad g(x) = x$

$x\ln(x) - \int_0^1 \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$

$x\ln(x) - \int_0^1 1 \, dx$

$[x\ln(x) - x]_0^1$

$[1 \cdot \ln(1) - 1] - \cancel{[0 \cdot \ln(0) - 0]}$

$\boxed{-1}$

Improper → limit → ?

$0 \cdot \infty$

hmm...

b. $J = \int_0^1 \ln(x)dx$

$f(x) = \ln(x) \quad f'(x) = \frac{1}{x}$

$g(x) = x \quad g'(x) = 1$

$= \ln(x) \cdot x - \int_0^1 \frac{1}{x}(x)\,dx$

should start with this

$= x\ln(x) - x \Big|_0^1 \longrightarrow \lim_{r \to 0^+} x\ln(x) - x \Big|_r^1$

$= (1\ln(1) - 1) - (r\ln(r) - r)$

$\quad -1 \qquad + +\infty$

$= \infty$

-2.5

b. $J = \int_0^1 \ln(x)dx$ $= \int_0^1 \ln(x) \cdot 1 \, dx$ Yes!?

$= [\ln(x) \cdot x]_0^1 - \int_0^1 \frac{1}{x} \cdot x \, dx$

$= -\int_0^1 1 \, dx = -[x]_0^1 = -1$

Improper → lim?

-3

c. $K = \int \frac{2x^2+5}{x^2-5x}dx$ $\rightarrow \int \left(2 + \frac{10x+5}{x^2-5x}\right) dx$

$$\begin{array}{r} 2 \\ x^2-5x \overline{)\,2x^2+5} \\ 2x^2-10x \\ \hline 10x+5 \end{array}$$

$= \int \left(2 + \frac{10x+5}{x(x-5)}\right) dx$

$* \frac{A}{x} + \frac{B}{x-5} = \frac{Ax - 5A + Bx}{x(x-5)} \Rightarrow A + B = 10$

$A = -1, \quad B = 11.$

$\Rightarrow \int \left(2 + \frac{-1}{x} + \frac{11}{x-5}\right) dx$

$= 2x - \ln|x| + 11 \ln|x-5| + C$

Problem 4 (10 pts): What is the partial fraction decomposition of $\frac{x-2}{(x^2+1)(x+1)^2}$?

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

	x	1
x	x^2	x
1	x	1

x^2+2x+1

	x^2	$2x$	1
Ax	Ax^3	$2Ax^2$	Ax
B	Bx^2	$2Bx$	B

$$\frac{(Ax+B)\overbrace{(x+1)^2}^{x^2+2x+1} + C\overbrace{(x+1)(x^2+1)}^{x^3+x^2+x+1} + D(x^2+1)}{(x^2+1)(x+1)^2} =$$

	x^2	1
x	x^3	x
1	x^2	1

x^3+x^2+x+1

$$= \frac{\cancel{Ax^3}+\cancel{Bx^2}+\cancel{2Ax^2}+\cancel{2Bx}+\cancel{Ax}+B+\cancel{Cx^3}+\cancel{Cx^2}+\cancel{Cx}+C+\cancel{Dx^2}+D}{(x^2+1)(x+1)^2}$$

$$= x^3(A+C) + x^2(B+2A+C+D) + x(2B+A+C) + B+C+D$$

$C=-A$

$A+C=0$

$2A+B+C+D=0$

$A+2B+C=1 \quad 2B=1 \quad \boxed{B=\tfrac{1}{2}}$

$B+C+D=-2$

$D=-2.5-C$

$-2.5+A$

$2(1)+\frac{1}{2}+{}^-1+D=0$

$\boxed{D=-1.5}$

$2A+\frac{1}{2}+{}^-A+{}^-2.5+A=0$

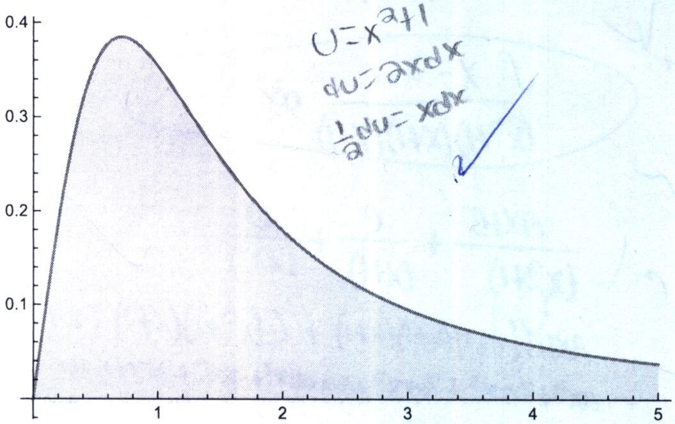
$2A=2$

$\boxed{A=1}$ $\boxed{C=-1}$

$$\boxed{\frac{x+\frac{1}{2}}{x^2+1}+\frac{-1}{x+1}+\frac{-1.5}{(x+1)^2}}$$

good work

Problem 5 (10 pts): Consider the infinite strip given by $0 \le y \le \frac{x}{(x^2+1)^{3/2}}$ for $0 \le x < \infty$ (part of which is shown below). Determine if it has finite area or not. If it does have finite area, find that area; if it doesn't have finite area, explain why not.



$U = x^2+1$

$dU = 2x\,dx$

$\frac{1}{2}dU = x\,dx$

$$\int_0^\infty \frac{x}{(x^2+1)^{3/2}}\,dx$$

$$\lim_{R\to\infty} \int_0^R \frac{x}{(x^2+1)^{3/2}}\,dx$$

$$\lim_{R\to\infty} \frac{1}{2}\int_{x=0}^{x=R} \frac{1}{U^{3/2}}\,dU$$

$$\lim_{R\to\infty} \frac{1}{2}\int_0^R U^{-3/2}\,dU$$

$$\lim_{R\to\infty} \frac{1}{2}\left(-2U^{-1/2}\right)\Big|_0^R$$

$$\lim_{R\to\infty} \frac{1}{\cancel{2}}\left(\frac{-\cancel{2}}{U^{1/2}}\right)\Big|_0^R$$

$$\lim_{R\to\infty} \frac{-1}{\sqrt{U}}\Big|_0^R$$

$$\lim_{R\to\infty} \frac{-1}{\sqrt{x^2+1}}\Big|_0^R$$

$$\lim_{R\to\infty} \frac{-1}{\sqrt{R^2+1}} + \frac{1}{\sqrt{0+1}}$$

$$\lim_{R\to\infty} \frac{-1}{\sqrt{R^2+1}} + 1$$

$$\lim_{R\to\infty} \frac{-1}{\sqrt{\infty^2+1}} + 1$$

$$\lim_{R\to\infty} \frac{-1}{\infty}^{0} + 1 = 1$$

finite area of 1 unit^2

good – x limits

Nice work

Problem 6 (15 pts): Compute the integral $I = \int_0^3 \frac{x^3\,dx}{\sqrt{25-x^2}}$ by trig-substitution.

$\sin(\theta) = \frac{x}{5}$

$x = 5\sin(\theta)$

$dx = 5\cos(\theta)\,d\theta$

$$\int_0^3 \frac{x^3\,dx}{\sqrt{25-x^2}} = \int_0^3 \frac{125\sin^3(\theta)}{5\sqrt{1-\sin^2(\theta)}}\,5\cos(\theta)\,d\theta$$

$$= 125\int_{0(x)}^{3(x)} \frac{\sin^3(\theta)}{\cos(\theta)}\cos(\theta)\,d\theta$$

$$= 125\int_{0(x)}^{3(x)} \sin^3(\theta)\,d\theta$$

$$= 125\int_{0(x)}^{3(x)} \sin(\theta)\sin^2(\theta)\,d\theta$$

$$= 125\int_{0(x)}^{3(x)} \sin(\theta)(1-\cos^2(\theta))\,d\theta$$

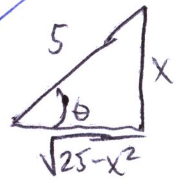
$u = \cos(\theta)$

$du = -\sin(\theta)\,d\theta$

$$= -125\int_{0(x)}^{3(x)} (1-u^2)\,du$$

$$= -125\left[u - \frac{u^3}{3}\right]_{0(x)}^{3(x)}$$

$$= -125\left[\cos(\theta) - \frac{\cos^3(\theta)}{3}\right]_{0(x)}^{3(x)}$$



$$= -125\left[\frac{\sqrt{25-x^2}}{5} - \frac{(\sqrt{25-x^2})^3}{375}\right]_0^3$$

$$= -125\left[\left(\frac{4}{5} - \frac{64}{375}\right) - \left(\frac{5}{5} - \frac{125}{375}\right)\right] = \boxed{\frac{14}{3}}$$

≈ 4.6667

good work