§8.1 HW #3, 10, 15

#3 $S = \mathbb{Z}$

$+$ by $x + y = \max(x, y)$

$\cdot$ by $x \cdot y = \min(x, y)$

a. Show that assoc., commutativity, distributivity hold. We're going to assume good knowledge of max & min:

$(x + y) + z = \max(\max(x, y), z) = \max(x, \max(y, z))$ ✓

$x + y = \max(x, y) = \max(y, x) = y + x$ ✓

$(x \cdot y) \cdot z = \min(\min(x, y), z) = \min(x, \min(y, z))$
$= x \cdot (y \cdot z)$ ✓

$x \cdot y = \min(x, y) = \min(y, x) = y \cdot x$ ✓

It's distributivity that's the pain... :-)

$x \cdot (y + z) = \min(x, \max(y, z))$ $\left(= x \cdot y + x \cdot z \text{ ; i.e. ?} \right.$
$\left. = \max(\min(x, y), \min(x, z)) \right)$

$\equiv$

[If $x \le y \lor x \le z$, then x;
if $\max(y, z) < x$, then $\max(y, z)$.]

$\equiv$

$= \max(\min(x, y), \min(x, z))$

$= x \cdot y + x \cdot z$

$\equiv$ - equivalent

$$x + (y \cdot z) = \max(x, \min(y,z))$$

$$\equiv \begin{bmatrix} \text{If } x \geq y \vee x \geq z, & x; \\ \text{if } \min(y,z) > x, & \min(y,z) \end{bmatrix}$$

$$\equiv \min(\max(x,y), \max(x,z)) \quad \text{Whew!}$$

$$= (x+y) \cdot (x+z)$$

But this is not a Boolean algebra, because it fails to have a special element 0. By contradiction: assume there is such an integer n,

Then $x+n = \max(x, n) = x$ for all integers. But $n-1$ is an integer, &

$$(n-1)+n = \max(n-1, n) = n \neq n-1.$$

So n fails to have the property.

$\therefore \nexists$ a zero, & we don't have a Boolean algebra.

#10 a, c, d Prove the following:

a. $(x+y) + (y \cdot x') = x+y$

$(x+y) + (y \cdot x') = (x+y+y) \cdot (x+y+x')$ Weird distributivity

$= (x+y) \cdot (x+x'+y)$ idempotence & commutativity

$= (x+y)\cdot(1+y)$ complements + "universal bound property" (p.624)

$= (x+y)\cdot 1$

$= x+y$ identity ✓

c. $(y'\cdot x) + x + (y+x)\cdot y' = x + (y'\cdot x)$

$(y'\cdot x) + x + (y+x)\cdot y' = y'\cdot x + x + y\cdot y' + x\cdot y'$ (dist)

$= x + y'\cdot x + 0 + y'\cdot x$ commutativity of + & $\cdot$, complements

$= x + y'\cdot x + y'\cdot x$ identity

$= x + y'\cdot x$ ✓ idempotence.

d. $(x+y')\cdot z = [(x'+z')\cdot(y+z')]'$

$[(x'+z')\cdot(y+z')]' = (x'+z')' + (y+z')'$ de Morgan

$= x\cdot z + y'\cdot z$ double neg., de Morgan

$= (x+y')z$ ✓ distributivity

#15 Define $\oplus$ by $x \oplus y = x\cdot y' + x'\cdot y$

Prove that

i) $x \oplus y = y \oplus x$ defn. commutativity $\cdot$ commutativity +

$= x\cdot y' + x'\cdot y = y'\cdot x + y\cdot x' = y\cdot x' + y'\cdot x$

$= y \oplus x$ ✓

ii) $x \oplus x = 0$ complements identity

$= x\cdot x' + x'\cdot x = 0 + 0 = 0$ ✓

iii) $0 \oplus x = x$

$$0 \oplus x = 0 \cdot x' + 0' \cdot x = 0 + 1 \cdot x$$ dual of universal bound property + identity

$$= x \checkmark$$ identities

$(\underline{0' = 1}$ - $0 + 0' = 1$ complements
$0' = 1$ identity$)$

iv) $1 \oplus x = x'$

$$= 1 \cdot x' + 1' \cdot x$$ identity

$$= x' + 0 \cdot x$$ complement (dual of)

$$= x' + 0$$ dual of universal bound property

$$= x' \checkmark$$ identity