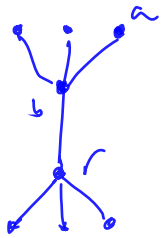
Section 6.2: Trees and their representations

March 20, 2020

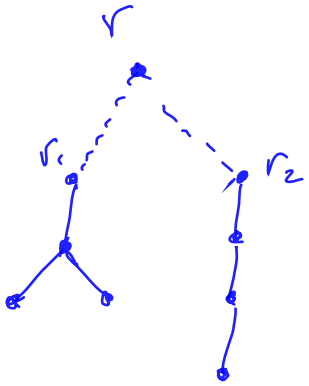
Abstract

Trees are defined, some applications are presented, some computer representation strategies are considered, and tree traversal algorithms are discussed. Trees are file cabinets: good ways to store stuff. Once it's stored, however, we'll need to retrieve it; hence we must be able to efficiently traverse the tree, checking what's in each node.



1 Tree Definition and Terminology

Definition: a **tree** is an acyclic, connected graph with one node designated as the **root** node. Note that trees are usually considered undirected, even though things tend to descend from the root....



"Because a tree is a connected graph, there is a path from the root to any other node in the tree; because the tree is acyclic, that path is unique [provided that no arc is used twice]." p. 510, Gersting.

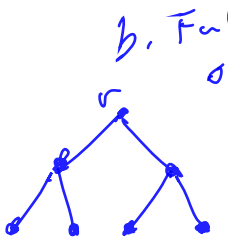
The set of trees can also be defined recursively, as follows:

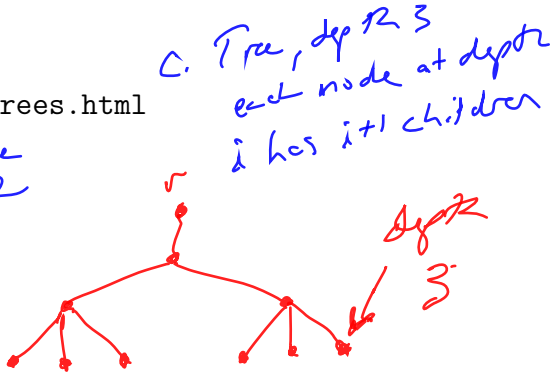
- **Base case:** A single node is a tree, with that node as root.
- **Inductive step:** The graph formed by attaching a new node r by a single arc to each root of trees $\{T_i\}$ is a tree.

Example: create a tree, subject to the definition. Some of you should make rather ordered trees; others might think of very strange trees.

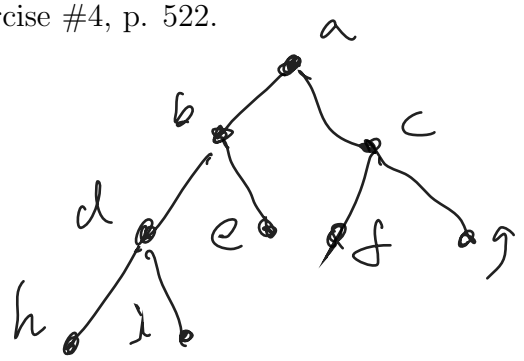
Use the tree terminology handout at
`http://www.nku.edu/~longa/classes/mat385_resources/docs/trees.html`
to classify your tree.

Consider Exercise #3, p. 521.





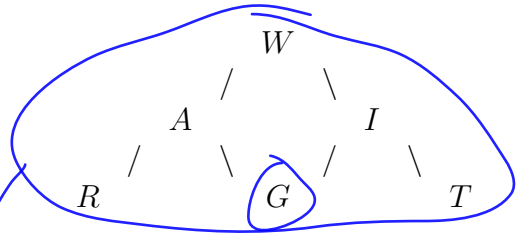
Exercise #4, p. 522.



a. binary? Yes
b. full binary? No
c. complete binary? Yes
d. Parent of e? b
e. right child of e? none
f. depth of g? 2
h. height of tree? 3

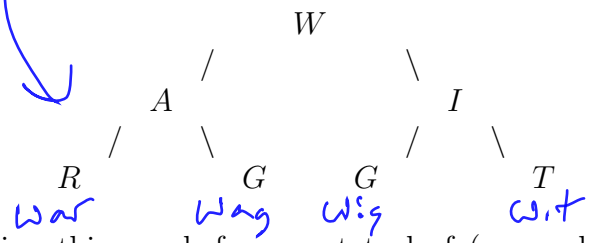
2 Examples of trees in action (p. 511)

- My favorite: the UNIX operating system (with root node called root!)
- Trees and tree-like graphs appear in funny places: I was listening to NPR (3/13/05) and Will Short, the Puzzle Guy, gave this puzzle:



This is a graph, but not a tree: it is connected, directed, and acyclic. Let's turn it into a tree, by duplicating some nodes:

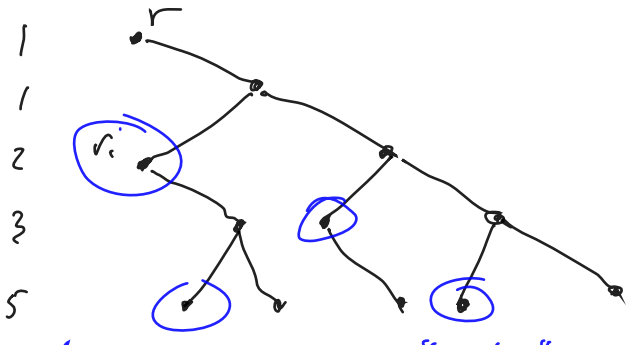
This is a tree! Each leaf a word.



In traversing this graph from root to leaf (more about this in a moment), in four possible ways, every way spells out a word: WAR, WAG, WIG, WIT; he asked us to create a full binary tree of depth 3 that uses the ten letters in ANALOGIES+K and does the same thing.... (as a hint, he said that an A is the left-most leaf, only not in those words – he used a lot more!:).

One of my students took the challenge of solving this (by brute force – oh well!;): Josh Werrmann

- Family "trees" (if intermarriages are forbidden!)
- Fibonacci numbers can be written as trees.



fractal tree:

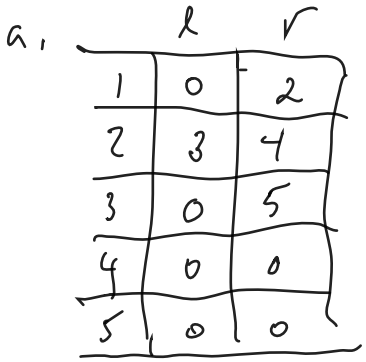
r_1 reproduces the tree of r, as does each new "pair".

- Org charts (most people report to only one person above them)

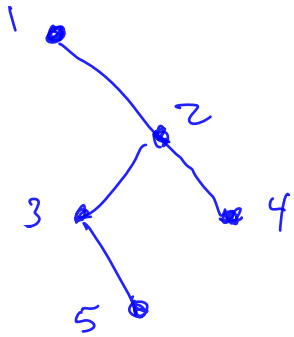
3 Tree Representations

Since trees are graphs, we can use the usual graph strategies for representation (e.g. matrices, adjacency lists).

Binary trees are special in that their children are classified as left and right. So in order to represent them, we need to specify left to right children for each node. Two representations are shown in Example 24, p. 513: a two-column array, and an adjacency list with three-field records. Notice how the adjacency list (pointer representation) is arranged to reproduce the geometry of the tree.

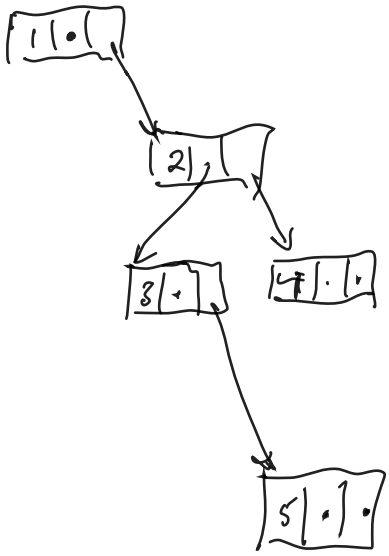


Practice 20, p. 513.



a. give left child - right child representation

b. Give the pointer representation



Notice that these representations (and the notion of left and right children) are **enhancements** of an ordinary graph: if we change the shape of the graph (without cutting arcs, adding nodes, etc.), then we should be looking at the same graph. But for a binary tree with right and left children, reflecting the graph would change the meaning (and the representation).

Trees are also different from graphs in that there is a special node – the root node.

4 Tree Traversal Algorithms

We'll look at three tree traversal algorithms, which are defined recursively – ah, recursion! – as follows (see p. 514):

preorder	*root*	*left*	*right*
inorder	*left*	*root*	*right*
postorder	*left*	*right*	*root*

Check out the animation showing how a tree is traversed by the three algorithms, at

`http://www.nku.edu/~longa/classes/mat385_resources/docs/traversal.htm`

It is clear from this animation how one handles non-binary tree traversal for pre- and postorder; for inorder, our book's algorithm would do "left, root, right, righter, ..., rightest"!

Note the nice definitions of these traversal algorithms based on the recursive definition of a tree. For example, our author's definition of algorithm preorder:

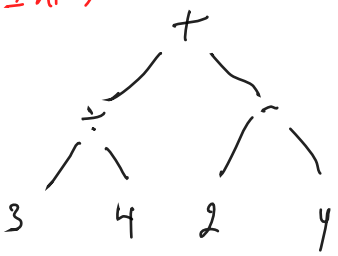
```
Preorder(tree T)
// Writes the nodes of a tree with root r in preorder
  write(r)
  for i=1 to t do
    Preorder(T_i)
  end for
end Preorder
```

t = # of children of T's root r, each a tree (T_i)

Exercise #21, p. 525 – traverse using each procedure

$3 \div 4 + (2 - y)$

Infix



prefix: + ÷ 3 4 - 2 y

postfix: 3 4 ÷ 2 y - +

Each traversal method has advantages in different situations. The example we consider is that of binary trees as representations of arithmetic expressions:

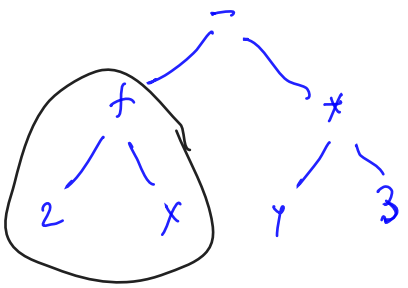
Example 23 and Figure 6.40, p. 512.

Boo for parentheses!

infix: (2+x) - (y*3)

prefix: - + 2 x * y 3

postfix: 2 x + y 3 * -



post-order allows deallocation of memory on the fly.

binary operations

- Inorder traversal writes the expressions in the most familiar form from our early schooldays (**Infix** notation). Inorder traversal requires the introduction of parentheses to make the meaning of the expression unique.

familiar with it

- Postorder traversal allows us to eliminate nodes once traversed (if we want to deallocate storage, for example - check, in the animation above, that you can deallocate each node as soon as it's been visited!). This is called Reverse Polish Notation (`http://www.hpmuseum.org/rpn.htm`), in honor of the Pole Jan Lukasiewicz.

maybe reverse honor?

"**Why Did/Does HP Use RPN?**

In the years that followed, computer scientists realized that RPN or postfix notation was very efficient for computer math. As a

postfix expression is scanned from left to right, operands are simply placed into a last-in, first-out (LIFO) stack and operators may be immediately applied to the operands at the bottom of the stack. By contrast, expressions with parentheses and precedence (infix notation) require that operators be delayed until some later point. Thus, the compilers on almost all modern computers converted statements to RPN for execution. (In fact, some computer manufacturers designed their computers around postfix notation.)"[1]

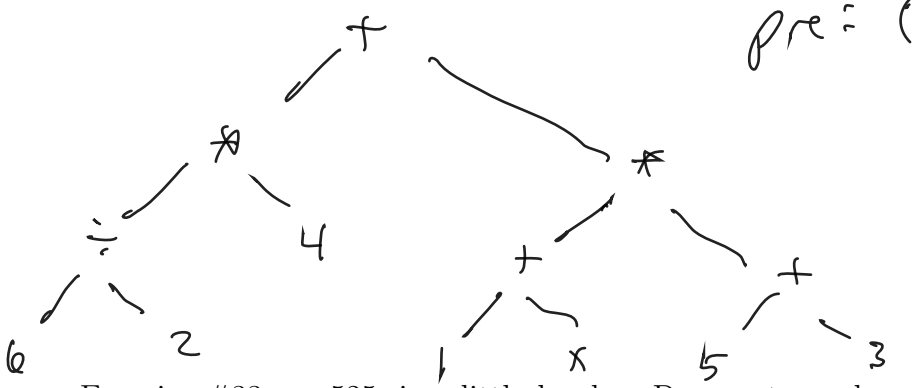
- Preorder traversal represents the expressions in Polish notation (which is what LISP uses (my favorite language)). Like postorder, preorder doesn't require parentheses (provided the operators are binary). If not all operations are binary, then parentheses creep in, as LISP lovers (and haters!) know only too well....

Polish notation!

Exercise #8, p. 522: draw the expression tree of

$$[(6 \div 2) * 4] + [(1 + x) * (5 + 3)]$$

(and write the expression in prefix and postfix notation).



pre: (+(*(÷6 2) 4)
(*(+ 1 x) (+ 5 3))
)

post: 6 2 ÷ 4 *
1 x + 5 3 + * +

Exercise #33, p. 525, is a little harder: Draw a tree whose preorder traversal is

$$a, b, c, d, e$$

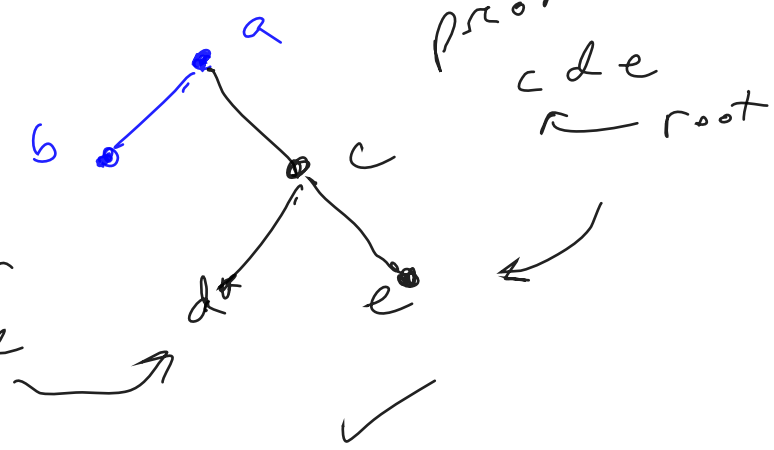
root first, so a is root

and whose inorder traversal is

$$b, a, d, c, e$$

So b is to the left

This one requires a little imagination!



preorder
c d e
↖ root

inorder
d c e

Trees may not be unique!
(Mathematicians hate that! ☺)

[1] From the link above, on `http://www.hpmuseum.org/rpn.htm`