Section 1.4: Predicate Logic

January 27, 2020

Abstract

We now consider the logic associated with predicate wffs, including a new set of derivation rules for demonstrating validity (the analogue of tautology in the propositional calculus) – that is, for proving theorems!

1 Derivation rules

- First of all, all the rules of propositional logic still hold. Whew! Propositional wffs are simply boring, variable-less predicate wffs.

- Our author suggests the following "general plan of attack":

  - strip off the quantifiers

  - work with the separate wffs

  - insert quantifiers as necessary

  Now, how may we legitimately do so? Consider the classic syllogism:

  a. (All) Humans are mortal.
  b. Socrates is human.

  Therefore Socrates is mortal.

  The way we reason is that the rule "Humans are mortal" applies to the specific example "Socrates"; hence the Socrates is mortal. We might write this as

  a. $(\forall x)\,(H(x) \to M(x))$
  b. $H(s)$

  Therefore $M(s)$ – it seems so obvious! But how do we justify that in a proof sequence?

- New rules for predicate logic: in the following, you should understand by the symbol x in $P(x)$ an expression with free variable x, possibly containing other (quantified) variables: e.g.

$$P(x) \equiv (\forall y)(\exists z)Q(x, y, z) \tag{1}$$

  - **Universal Instantiation**: from $(\forall x)P(x)$ deduce $P(t)$.

    Caveat: t must not already appear as a variable in the expression for $P(x)$: in the equation above, (1), it would not do to deduce $P(y)$ or $P(z)$, as those variables appear in the expression (in a quantified fashion) already.

    Example: Practice 22, p. 60. Prove:

    $$(\forall x)[P(x) \to R(x)] \wedge [R(y)]' \to [P(y)]'$$

    1. $(\forall x)[P(x) \to R(x)]$ hyp
    2. $R(y)'$ hyp
    3. $P(y) \to R(y)$ 1, ui
    4. $P(y)'$ 2, 3, mt

  - **Existential Instantiation**: from $(\exists x)P(x)$ deduce $P(t)$.

    Caveat: t must be introduced for the first time (so do these early in proofs). You can do a universal instantiation which also uses t after an existential instantiation with t, but not *vice versa* (e.g. Example 27, p. 60).

    Example: Ex. #12, p. 70 (start). Prove that the following wff is a valid argument:

    $$(\forall x)P(x) \wedge (\exists x)Q(x) \to (\exists x)[P(x) \wedge Q(x)]$$

    1. $(\forall x)P(x)$ hyp
    2. $(\exists x)Q(x)$ "
    3. $Q(t)$ 2 ei
    4. $P(t)$ 1, ui
    5. $P(t) \wedge Q(t)$ 3, 4 conj

    existential before universal (almost always)

– **Universal Generalization**: from $P(x)$ deduce $(\forall x)P(x)$.

Caveats:

* $P(x)$ hasn't been deduced by existential instantiation from any hypothesis in which x was free (p. 63, top), and
* $P(x)$ hasn't been deduced by existential instantiation from another wff in which x was free. For example, suppose that we wanted to prove, in the domain of the integers, that:

$$(\forall x)(\exists y)(x+y=0) \to (\forall x)(x+a=0)$$

1.$(\forall x)(\exists y)(x+y=0)$	*hyp*
2.$(\exists y)(x+y=0)$	1, *ui*
3.$x+a=0$	2, *ei*
4.$(\forall x)(x+a=0)$	3, **incorrect ug**

Example: Ex. #20, p. 71. Prove:

$$(\forall x)[P(x) \to Q(x)] \to [(\forall x)P(x) \to (\forall x)Q(x)]$$

1. $(\forall x)[P(x) \to Q(x)]$ hyp
2. $(\forall x)P(x)$ ded. method
3. $P(x)$ 2, ui
4. $P(x) \to Q(x)$
5. $Q(x)$ 3, 4 mp
6. $(\forall x)Q(x)$ 5, ug

(Note: the deduction method still applies, of course.)

– **Existential Generalization**: from $P(a)$ deduce $(\exists x)P(x)$.

Caveat: x must not appear in $P(a)$.

Example: Ex. #12, p. 70 (finish).

5. $P(t) \wedge Q(t)$
6. $(\exists x)(P(x) \wedge Q(x))$ 5, e.g.

2 Some results/notes

– Note that

$$(\forall y)[P(x) \to Q(x,y)] \iff [P(x) \to (\forall y)Q(x,y)]$$

as shown on pp. 64, and

$$(\exists y)[P(x) \to Q(x,y)] \iff [P(x) \to (\exists y)Q(x,y)]$$

as Gersting suggests (bottom, p. 64). How would we demonstrate that? (See "temporary hypothesis", below.)

This means that we can "pass over" predicates outside our own scope, or include them within our own scope – provided they do not conflict with other similarly named variables. This is similar to what we do with summation notation, when, for example, we can write

$$\sum_{i=1}^{m}\sum_{j=1}^{n} A(i)B(j) = \sum_{i=1}^{m} A(i) \sum_{j=1}^{n} B(j)$$

– Note also the **method** of proof: the author introduces a **temporary hypothesis**. If you think about the deduction method, it takes a conclusion which is an implication and rewrites it so that the implication disappears (the antecedent becomes one of the hypotheses). Similarly, we can take an hypothesis (in this case, one which we introduce) and turn a conclusion into an implication. This is the deduction method backwards! That is, suppose that one starts with $P(x)$ as true. Suppose further that if you add a "temporary hypothesis" $Q(x)$ then you can deduce $R(x)$:

$$P(x) \wedge Q(x) \to R(x)$$

Using the deduction method backwards, we conclude that

$$P(x) \to (Q(x) \to R(x))$$

Since $P(x)$ implies the implication $Q(x) \to R(x)$, we can add it as an hypothesis to our argument:

$$P(x) \wedge (Q(x) \to R(x))$$

Think about it....

Here's a simple example of how it works. Suppose that I am the king. Now suppose that, in the same kingdom, there is a queen (ruling in the kingdom). Then she must be my wife, by the laws of royalty. Hence, to the proposition "There is a king" we could attach the hypothesis "If there is a queen (ruling in the kingdom), then the queen is the king's wife." This additional hypothesis is true as long as there is a king (which is hypothesized). We've got the following chain of events:

$$\begin{array}{cc} K & \textit{hypothesis} \\ Q & \textit{temp. hyp.} \\ W(Q,K) & \textit{laws of royalty} \\ Q \to W(Q,K) & \textit{temp. hyp. discharged} \end{array}$$

Note that this "discharged" hypothesis is only true in the context of the hypotheses already assumed

$$K \wedge (Q \to W(Q,K))$$

(if there were no king, it might be that there is a queen who is not the king's wife – e.g. Queen Elizabeth I).

Look at the three proofs using a temporary hypothesis (Examples #31, and 32(a,b)). Notice how the introduction of the temporary hypothesis ends with an implication, which is then useful for the continuation of the proof.

Example: Practice 25, p. 65: Prove:

$$(\forall x)[(B(x) \vee C(x)) \to A(x)] \to (\forall x)[B(x) \to A(x)]$$

1. $(\forall x)[B(x) \vee C(x) \to A(x)]$ hyp
2. $B(x) \vee C(x) \to A(x)$ 1, ui
3. $B(x)$ temp hyp
4. $B(x) \vee C(x)$ 3, add
5. $A(x)$ 2, 4, mp
6. $B(x) \to A(x)$ temp hyp discharged
7. $(\forall x)[B(x) \to A(x)]$

So now, how would we demonstrate that

$$(\exists y)[P(x) \to Q(x,y)] \iff [P(x) \to (\exists y)Q(x,y)]$$

(by the way, I really shudder when I see this one – this is a technical argument, that shows that it's **possible** to have $\exists$ and $\to$ together....

Example: #39, p. 72 Every computer science student works harder than somebody, and everyone who works harder than any other person gets less sleep than that person. Maria is a computer science student. Therefore, Maria gets less sleep than someone else. $C(x), W(x,y), S(x,y), m$