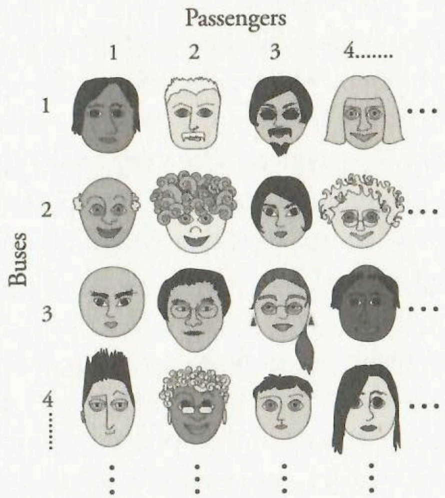
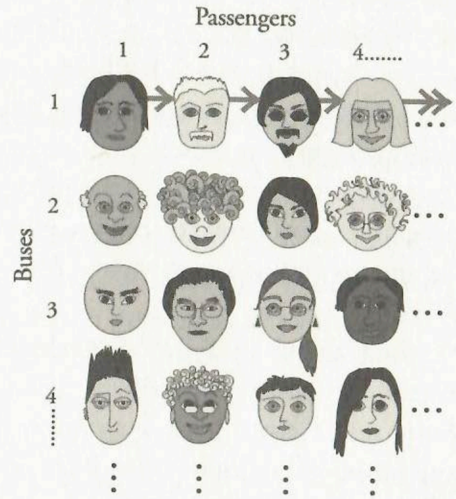


Of course, we can't show literally all of them here, since the diagram would need to be infinite in both directions. But a finite version of the picture is adequate. The point is that any *specific* bus passenger (your aunt Inez, say, on vacation from Louisville) is sure to appear on the diagram somewhere, as long as we include enough rows and columns. In that sense, everybody on every bus is accounted for. You name the passenger, and he or she is certain to be depicted at some finite number of steps east and south of the diagram's corner.

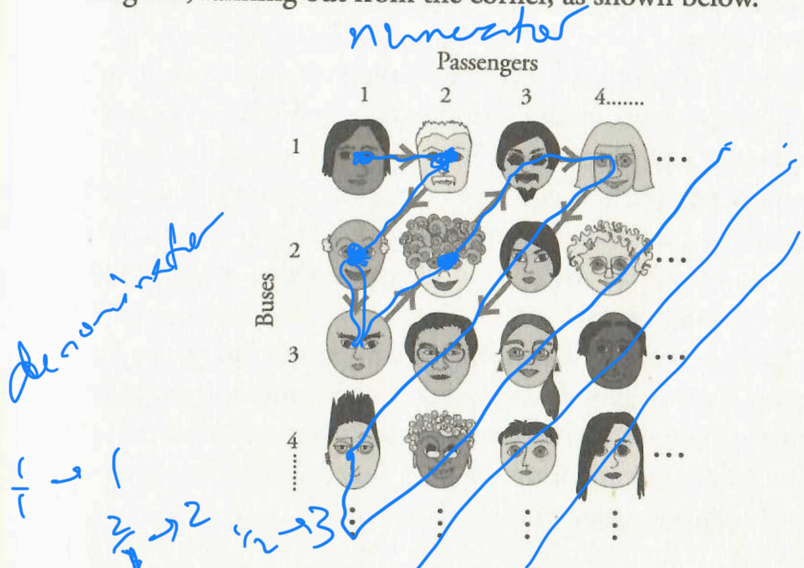
The manager's challenge is to find a way to work through this picture systematically. He needs to devise a scheme for assigning rooms so that everybody gets one eventually, after only a *finite* number of other people have been served.

Sadly, the previous manager hadn't understood this, and mayhem ensued. When a similar convoy showed up on his watch, he became so flustered trying to process all the people on bus 1 that he never got around to any other bus, leaving all those neglected passengers screaming and furious. Illustrated

on the diagram below, this myopic strategy would correspond to a path that marched eastward along row 1 forever.



The new manager, however, has everything under control. Instead of tending to just one bus, he zigs and zags through the diagram, fanning out from the corner, as shown below.



He starts with passenger 1 on bus 1 and gives her the first empty room. The second and third empty rooms go to passenger 2 on bus 1 and passenger 1 on bus 2, both of whom are depicted on the second diagonal from the corner of the diagram. After serving them, the manager proceeds to the third diagonal and hands out a set of room keys to passenger 1 on bus 3, passenger 2 on bus 2, and passenger 3 on bus 1.

I hope the manager's procedure—progressing from one diagonal to another—is clear from the picture above, and you're convinced that any particular person will be reached in a finite number of steps.

So, as advertised, there's always room at the Hilbert Hotel.

The argument I've just presented is a famous one in the theory of infinite sets. Cantor used it to prove that there are exactly as many positive fractions (ratios p/q of positive whole numbers p and q) as there are natural numbers (1, 2, 3, 4, . . .). That's a much stronger statement than saying both sets are infinite. It says they are infinite to precisely the same extent, in the sense that a one-to-one correspondence can be established between them.

You could think of this correspondence as a buddy system in which each natural number is paired with some positive fraction, and vice versa. The existence of such a buddy system seems utterly at odds with common sense—it's the sort of sophistry that made Poincaré recoil. For it implies we could make an exhaustive list of all positive fractions, even though there's no smallest one!

And yet there is such a list. We've already found it. The fraction p/q corresponds to passenger p on bus q, and the argument above shows that each of these fractions can be paired off with a certain natural number 1, 2, 3, . . . , given by the passenger's room number at the Hilbert Hotel.

The coup de grâce is Cantor's proof that some infinite sets are bigger than this. Specifically, the set of real numbers between 0 and 1 is uncountable—it can't be put in one-to-one correspondence with the natural numbers. For the hospitality industry, this means that if all these real numbers show up at the reception desk and bang on the bell, there won't be enough rooms for all of them, even at the Hilbert Hotel.

The proof is by contradiction. Suppose each real number could be given its own room. Then the roster of occupants, identified by their decimal expansions and listed by room number, would look something like this:

ℕ real #s between 0 and 1

Room 1: .6708112345 . . .
Room 2: .1918676053 . . .
Room 3: .4372854675 . . .
Room 4: .2845635480 . . .

Remember, this is supposed to be a complete list. Every real number between 0 and 1 is supposed to appear somewhere, at some finite place on the roster.

Cantor showed that a lot of numbers are missing from any such list; that's the contradiction. For instance, to construct one that appears nowhere on the list shown above, go down the diagonal and build a new number from the underlined digits:

Room 1: .6708112345 . . .
Room 2: .1918676053 . . .
Room 3: .4372854675 . . .
Room 4: .2845635480 . . .

one-to-one

.5137 . . .

The decimal so generated is .6975 . . .

But we're not done yet. The next step is to take this decimal and change all its digits, replacing each of them with any *other* digit between 1 and 8. For example, we could change the 6 to a 3, the 9 to a 2, the 7 to a 5, and so on.

This new decimal .325 . . . is the killer. It's certainly not in room 1, since it has a different first digit from the number there. It's also not in room 2, since its second digit disagrees. In general, it differs from the *n*th number in the *n*th decimal place. So it doesn't appear anywhere on the list!

The conclusion is that the Hilbert Hotel can't accommodate all the real numbers. There are simply too many of them, an infinity beyond infinity.

And with that humbling thought, we come to the end of this book, which began with a scene in another imaginary hotel. A *Sesame Street* character named Humphrey, working the lunch shift at the Furry Arms, took an order from a roomful of hungry penguins—"Fish, fish, fish, fish, fish, fish"—and soon learned about the power of numbers.

It's been a long journey from fish to infinity. Thanks for joining me.

$\mathcal{P}(\mathbb{R}) > \mathbb{R}$

$\mathcal{P}(\mathcal{P}(\mathbb{R})) > \mathcal{P}(\mathbb{R})$

There is an infinite ladder of bigger & bigger infinities....

$1.00000000\ldots = .99999999\ldots$

Acknowledgments

$\mathcal{P}(\mathbb{N})$ we can show is a bigger infinity than $\mathbb{N}$. Is it the same size as R (the real numbers)?

Many friends and colleagues helped improve this book by generously offering their sage advice—mathematical, stylistic, historical, and otherwise. Thanks to Doug Arnold, Sheldon Axler, Larry Braden, Dan Callahan, Bob Connelly, Tom Gilovich, George Hart, Vi Hart, Diane Hopkins, Herbert Hui, Cindy Klauss, Michael Lewis, Michael Mauboussin, Barry Mazur, Eri Noguchi, Charlie Peskin, Steve Pinker, Ravi Ramakrishna, David Rand, Richard Rand, Peter Renz, Douglas Rogers, John Smillie, Grant Wiggins, Stephen Yeung, and Carl Zimmer.

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