Exam 3

MAT 229, Spring 2021

Show your work to receive credit

A. Analyze each of the given series. If a series diverges, give reasons why; if it converges, either

- give the exact value it converges to, or
- provide an error estimate in approximating the sum with the 10th partial sum.

1. $\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k$

This is a geometric series with $a = 1$ and $r = \frac{2}{3}$, which means it converges to $\frac{1}{1-2/3} = 3$.

2. $\sum_{k=1}^{\infty} \frac{k}{30\,k-20}$

Applying the divergence test, $\lim_{k\to\infty} \frac{k}{30\,k-20} = \frac{1}{30} \neq 0$. By the divergence test the series diverges.

3. $\sum_{k=1}^{\infty} \frac{2}{k^3}$

This is a p-series with $p = 3$, so it converges. Applying the integral test error estimate the approximation $\sum_{k=1}^{10} \frac{2}{k^3} \approx \sum_{k=1}^{\infty} \frac{2}{k^3}$ has

error $\leq \int_{10}^{\infty} \frac{2}{k^3}\, dk = -\frac{1}{k^2} \mid_{10}^{\infty} = 0 + \frac{1}{100} = 0.01$

4. $\sum_{k=0}^{\infty} e^{-2\,k}$

This is a geometric series with $r = e^{-2} \approx 1.35$, so it converges to $\frac{1}{1-e^{-2}} \approx 1.15652$.

Alternatively, use the integral test on it to see

$\int_0^{\infty} e^{-2\,k}\, dk = -\frac{1}{2} e^{-2\,k} \mid_0^{\infty} = 0 + 1/2$

Since the improper integral converges, so to does the series. Applying the integral test error estimate, the approximation $\sum_{k=0}^{10} e^{-2\,k}$ has error $\leq \int_{10}^{\infty} e^{-2\,k}\, dk = -\frac{1}{2} e^{-2\,k} \mid_{10}^{\infty} = 0 + 1/2\, e^{-20} \approx 1.03 \times 10^{-9}$

5. $\sum_{k=0}^{\infty} (-1)^k \frac{3}{k+1}$

Applying the alternating series test with $a_k = \frac{3}{k+1}$:

1. $a_k = \frac{3}{k+1} \geq 0$
2. $a_{k+1} \leq a_k$, since $\frac{3}{k+1+1} \leq \frac{3}{k}$
3. $\lim_{k\to\infty} \frac{3}{k+1} = 0$

So this series converges. Applying the alternating series test, the approximation $\sum_{k=0}^{10} (-1)^k \frac{3}{k+1}$ has error $\leq \frac{3}{11+1} = 0.25$.

6. $\sum_{k=53}^{\infty} \frac{k}{k^2-1}$

Using the limit comparison test, for large values of k $\frac{k}{k^2-1} \approx \frac{1}{k}$ and $\sum_{k=53}^{\infty} \frac{1}{k}$ diverges. Therefore, so does the original.

B. Determine whether the following improper integrals converge or diverge (with reasons):

7. $\int_0^1 \frac{1}{x^2+x^{1/2}}\,dx$

Since $\frac{1}{x^2+x^{1/2}} \le \frac{1}{x^{1/2}}$, $\int_0^1 \frac{1}{x^2+x^{1/2}}\,dx \le \int_0^1 \frac{1}{x^{1/2}}\,dx = 2\,x^{1/2}\,\big|_0^{1/2} = \sqrt{2}$. Since this integral converges, so must the original converge.

8. $\int_1^\infty \frac{\ln(x)}{x}\,dx$

Using the substitution $u = \ln(x)$, $du = \frac{1}{x}\,dx$ this improper integral becomes

$$\int_1^\infty \frac{\ln(x)}{x}\,dx = \int_0^\infty u\,du = \frac{u^2}{2}\,\big|_0^\infty = \infty$$

This improper integral diverges.

C. For $f(x) = e^{-2x}$:

9. Determine the 3rd Taylor polynomial T_3 for f about $x = 0$.

Since $f(x) = e^{-2x}$, $f^{(n)}(x) = (-2)^n\,e^{-2x}$ and $f^{(n)}(0) = (-2)^n$. The third degree Taylor polynomial is

$$T_3(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}\,x^2 + \frac{f'''(0)}{6}\,x^3 = 1 - 2\,x + 2\,x^2 - \frac{4}{3}\,x^3$$

10. Use the remainder term R_3 to provide a bound on the error you might make approximating f by T_3 on the interval [–1, 1].

Since $|\,R_3(x)\,| \le \frac{K\,|\,x\,|^4}{4!}$ where $K \ge \left|f^{(4)}(x)\right| = 16\,e^{-2x}$ for x in [–1, 1]. Here choose $K = 16\,e^{-2\,(-1)} = 16\,e^2 \approx 118.23$. Then

$$|\,R_3(x)\,| \le \frac{K\,|\,x\,|^4}{4!} \le \frac{118.23\,(1)}{24} = 4.93$$