Section Summary: Applications of Taylor Polynomials

1 Definitions

Taylor series of f about a (assuming f has derivatives of all orders):

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The n^{th}**-degree Taylor polynomial of f at** a:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Then $R_n(x) = f(x) - T_n(x)$ is the called the **remainder** of the Taylor series.

We use T_n to approximate f, especially in the vicinity of $x = a$: $f(x) \approx T_n(x)$.

2 Applications

2.1 Taylor nails the tangent line!

We first observe that the first degree Taylor polynomial about $x = a$ is simply the tangent line (linearization) to f at $x = a$:

$$T_1(x) = f(a) + f'(a)(x-a)$$

We can then think of extending the idea of a tangent line to "tangent parabola", "tangent cubic", etc.

2.2 Computation of functions in your calculator or computer

There are four things that your computer or calculator knows how to do well: add, subtract, multiply, and divide. Polynomials need only those four operations. But how do we compute $\sin(x)$? Can we to teach our computers SOHCAHTOA, and how to draw right triangles?

No, but we can fake a good sine with Taylor polynomials. So, for example,

$$\sin(x) \approx x$$

when x is small. This approximation is used by physicists in the pendulum (harmonic oscillator) problem, for example, as a **first-order** approximation.

$$\sin(x) \approx T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

is an even better approximation, of course (and I imagine that you see the pattern if you'd like to continue).

We can then use periodicity and symmetry to complete the picture:

$$sinFake(x) = \begin{cases} -sinFake(-x) & x < 0 \\ T_7(x) & x \leq \frac{\pi}{2} \\ sinFake(x - floor\left(\frac{x}{2\pi}\right)2\pi) & x > 2\pi \\ -sinFake(2\pi - x) & x > \pi \\ sinFake(\pi - x) & x > \frac{\pi}{2} \end{cases}$$

Of course, once you have a fake sine, you have a fake cosine, a fake tangent, a fake csc, etc.....

Question: could you create a fake arcsin from this?

2.3 Applications to physics

We've already mentioned the differential equation for the harmonic oscillator, which looks like

$$x''(t) + \sin(x(t)) = 0$$

This is approximately equal to

$$x''(t) + x(t) = 0$$

for small values of x (the displacement), which has for solutions $x(t) = a\sin(t) + b\cos(t)$.

Another application is in the approximation to the relativistic equation for the mass m of an object with rest mass m_o:

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}}$$

The Kinetic energy of an object of rest mass m_o is $K = mc^2 - m_oc^2$. Show that when v is small relative to speed of light c, then $K \approx \frac{1}{2}m_ov^2$ (the classical kinetic energy).

In other words, at "ordinary speeds" we humans deal with, the relativistic equations reduce to the Newtonian equations. So

$$K = \frac{m_oc^2}{\sqrt{1 - v^2/c^2}} - m_oc^2 = m_oc^2\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right) = m_oc^2\left[\left(\frac{c^2 - v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right]$$

That is,

$$K = m_oc^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right]$$

Now we're going to replace that expression for K with a Taylor series,

$$K = m_oc^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right) - 1\right]$$

and then approximate on the right with a Taylor polynomial:

$$K \approx m_oc^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) - 1\right] = m_oc^2\left[\frac{1}{2}\frac{v^2}{c^2}\right] = \frac{1}{2}m_ov^2$$

as we thought. The error we're making is on the order of $\frac{v^4}{c^2}$, which is incredibly small for any "ordinary" speeds we deal with. So Newton had every good reason to think that he was exactly right. Little did he know.... Well, actually, he knew a fair amount; and so did Einstein!