Section Summary: Series

1 Definitions

An **infinite series** is the sum of the terms of an infinite sequence,

$$\sum_{n=1}^{\infty} a_n$$

We determine whether it converges or not by adding up finitely many terms in what is known as a partial sum. We can add 1, 2, 3, ..., and n terms, and so on. This process generates a *sequence* of partial sums, and hence we can talk about the convergence of that sequence. If the sequence of partial sums has a limit, then the infinite series is convergent.

Given a series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \ldots$. Let s_n denote its n^{th} partial sum:

$$s_n = \sum_{i=1}^{n} a_i$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ is a real number, then the series $\sum_{i=1}^{\infty} a_i$ is convergent, and we write

$$\sum_{i=1}^{\infty} a_i = s$$

The number s is called the **sum** of the series. Otherwise, the series is called **divergent**.

One famous series is the **geometric series**, which is defined as

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$$

Another famous series is the **harmonic series**, which is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

2 Theorems

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$, with sum

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

It is divergent otherwise.

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. By contrast then, if $\lim_{n\to\infty} a_n$ fails to exist, or exists but is different from 0, then the series diverges. Unfortunately, however, $\lim_{n\to\infty} a_n = 0$ is not enough to guarantee that the series converges. An example is the harmonic series.

3 Properties, Hints, etc.

If $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant, then so are

$$\begin{array}{l} \sum(a_n + b_n) = \sum a_n + \sum b_n \\ \sum(a_n - b_n) = \sum a_n - \sum b_n \\ \sum ca_n = c \sum a_n \end{array}$$

One method for showing that a series is divergent is by comparison with some known divergent series (e.g. the harmonic series).

4 Summary

Series are important, but exceedingly odd. What does it mean to add up an infinite number of things? It is a bizarre notion, which troubled Greeks like Zeno, who used the confusion to prove that motion is possible and impossible, that a tortoise can always beat a hare in a race, etc.

We think that we've got a handle on it now, thanks to the ideas of partial sums, and convergence of sequences. I hope that we do!