Indeterminate Forms and L'Hôpital's rule

a. Definitions

An **indeterminate form** is one that requires further study to be evaluated. One of the most important in calculus is the form of type $\frac{0}{0}$, since the derivative function limit is of this form:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If f is continuous at x, then this form is indeterminate of type $\frac{0}{0}$.

Other indeterminate forms include $\frac{\infty}{\infty}$, e.g.

$$\lim_{x \to \infty} \frac{\ln(x)}{x-1}$$

and $0 * \infty$, e.g.

$$\lim_{x \to 0} x \ln(x).$$

b. Theorems

- **L'Hôpital's rule**: Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

  $$\lim_{x \to a} f(x) = 0 \;\text{ and }\; \lim_{x \to a} g(x) = 0$$

  or that

  $$\lim_{x \to a} f(x) = \pm\infty \;\text{ and }\; \lim_{x \to a} g(x) = \pm\infty$$

  Then

  $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

  If the limit on the right side exists (or is ∞ or $-\infty$).
  These quotients are called indeterminate forms, since we can't tell which way they'll turn out by inspection.

c. Properties/Tricks/Hints/Etc.

- Other indeterminate forms include products $0 * \infty$, which we can rewrite as quotients: e.g.

$$\lim_{x\to 0} \sin(x)\ln(x) = \lim_{x\to 0} \frac{\ln(x)}{\csc(x)}$$

- Differences can be indeterminate, e.g. $\infty - \infty$, which we may also be able to rewrite as quotients: e.g.

$$\lim_{x\to\infty} (x - \sqrt{x^2 - x}) = \lim_{x\to\infty} \frac{(x - \sqrt{x^2 - x})(x + \sqrt{x^2 - x})}{(x + \sqrt{x^2 - x})}$$

so

$$\lim_{x\to\infty} (x - \sqrt{x^2 - x}) = \lim_{x\to\infty} \frac{(x^2 - (x^2 - x))}{(x + \sqrt{x^2 - x})} = \lim_{x\to\infty} \frac{x}{(x + \sqrt{x^2 - x})}$$

which we see is an indeterminate form as a quotient.

- If

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

is indeterminate, then you can apply the rule again (provided the derivative functions satisfy the constraints of the theorem):

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

etc. This could go on for a long time!

d. Summary

There are certain limits that are "indeterminate", and L'Hôpital's rule provides us a method for determining them. Especially important are limits that are essentially of the form $\infty * 0$, or $\frac{0}{0}$, or $\frac{\infty}{\infty}$.