Lab 13

MAT 229, Spring 2021

Review

The **n^{th} degree Taylor polynomial of $f(x)$ centered at a** is the partial sum of the Taylor series that goes up to and includes the n^{th} power of $(x - a)$. If the Taylor series for $f(x)$ centered at a converges to $f(x)$ for a given value x, then the n^{th} Taylor polynomial of $f(x)$ centered at a provides a polynomial approximation to $f(x)$.

The **n^{th} Taylor remainder of $g(x)$ centered at a** is

$$R_n(x) = g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(x - a)^k$$

In other words the error in using the n^{th} degree Taylor polynomial to approximate the function is error = $|R_n(x)|$

Analyzing error

The Taylor series error estimate: If $\left| f^{(n+1)}(x) \right| \leq M$ for all values of x of interest, then

$$\left| R_n(x) \right| \leq \frac{M}{(n+1)!} \left| x - a \right|^{n+1}.$$

You can think of this roughly as the error on the interval is smaller than the largest "first neglected term" on the interval.

Questions to submit

Instructions: Do your work on paper and submit as a pdf file. **Show your work.**

1. Let $g(x) = \sin(x)$.

- What is the Taylor series centered at zero for $g(x)$? $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$
- What is a simple estimate for M in the remainder for this function? ($\left| g^{(n+1)}(x) \right| \leq M$) $M = 1$
- What degree Taylor polynomial for $g(x)$ will approximate it with error less than 0.01 for $|x| < 1$? 4
- What degree Taylor polynomial for $g(x)$ will approximate it with error less than 0.01 for $|x| < 2$? 7
- What degree Taylor polynomial for $g(x)$ will approximate it with error less than 0.01 for $|x| < 3$? 10

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - 0|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!}$$

Each of these must be $< \frac{1}{100}$ $\quad \frac{1}{(n+1)!} \quad \frac{2^{n+1}}{(n+1)!} \quad \frac{3^{n+1}}{(n+1)!}$

```
In[308]:= nterms = 3 (* Why does nothing change if you replace 3 with 4? *)
          f[x_] := Sin[x]
          Series[f[x], {x, 0, nterms}]
          sn[x_] = Normal[Series[f[x], {x, 0, nterms}]];
          Plot[{f[x], sn[x]}, {x, -3, 3}]
          Plot[{f[x] - sn[x]}, {x, -3, 3}]
```

$\frac{1}{(4+1)!}$ $\quad n=4 \quad n=\cancel{4}\,6 \quad n=10$

$120 = 5!$

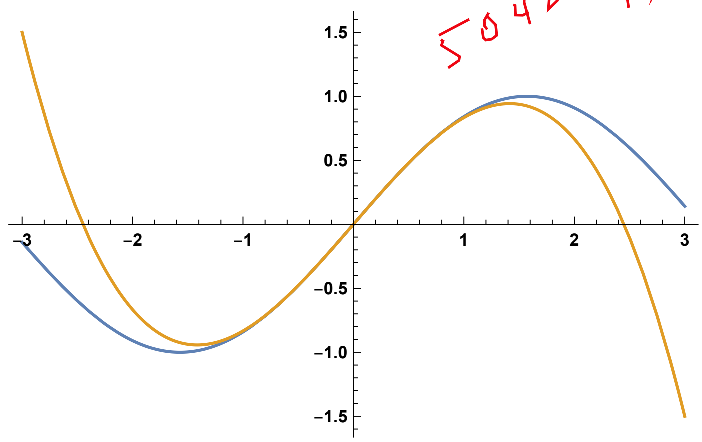
$720 = 6!$

$5040 = 7!$

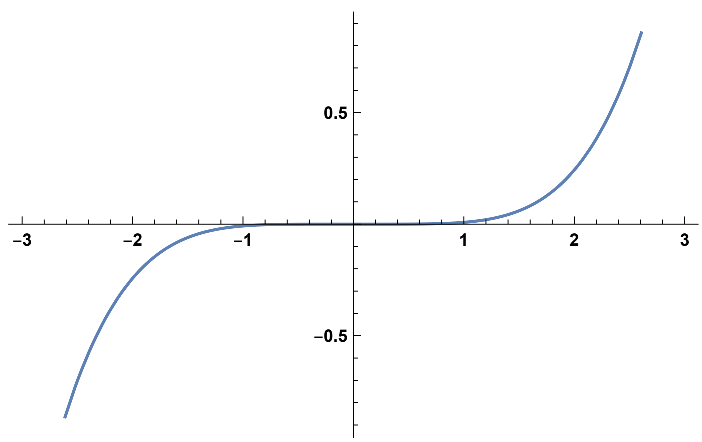
Out[308]= 3

Out[310]= $x - \frac{x^3}{6} + O[x]^4$

Out[312]=



Out[313]=



(Video)

2. Let $h(x) = \ln(x)$.

- What is the Taylor series centered at 1 for $h(x)$?

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$$

- What is the interval of convergence for that Taylor series?

(Video)

$[0,2]$

Scratch at left end

- What is true about any estimate for M in the remainder for this function for the values of x in the interval of convergence?

Converges at 2 by AST

ratio test:

$$\lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+2}(x-1)^{k+1}}{k+1}}{\frac{(-1)^{k+1}(x-1)^k}{k}} \right| = \lim_{k\to\infty} \left| \frac{k}{k+1}(x-1) \right| = |x-1| < 1$$

when $x \in (0,2)$

$5! \cdot 2^6$

$\leq 1/6$

- What if we use the 5^{th} degree Taylor polynomial, $T_5(x)$, centered at 1 to approximate $h(x)$ for $|x-1| \leq 0.5$? What is a good estimate for M?
- Approximate $\ln(1.5)$ with $T_5(1.5)$. What is an estimate of the error in this approximation?

$|h^{(6)}(x)|$

$M \geq |h^{(6)}(x)|$ on $[1/2, 3/2]$

$h^{(6)}(x) = \dfrac{(-1)^{k+1}(6-1)!}{x^6}$

largest when x is smaller

$M = \dfrac{5!}{(1/2)^6} = 5! \, 2^6$

$|R_5(1.5)| \leq \dfrac{5! \, 2^6}{(5+1)!}(1/2)^{5+1} = \dfrac{1}{6}$

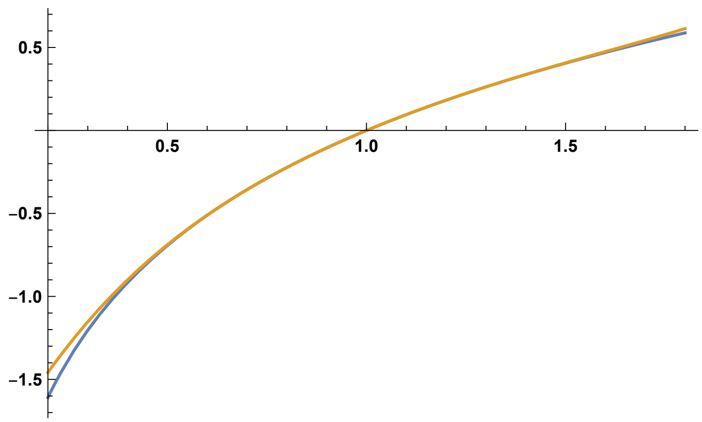
$\leq \dfrac{1}{6}$

In[314]:=
```
nterms = 5
f[x_] := Log[x]
Series[f[x], {x, 1, nterms}]
sn[x_] = Normal[Series[f[x], {x, 1, nterms}]];
Plot[{f[x], sn[x]}, {x, 0.2, 1.8}]
Plot[{f[x] - sn[x]}, {x, 0.2, 1.8}, PlotLabel → "Absolute Error"]
```

Out[314]= 5

Out[316]= $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 + O[x-1]^6$

Out[318]=



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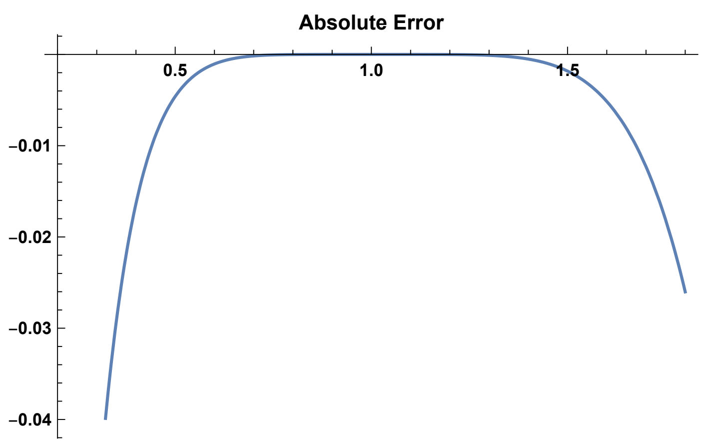


(Video)

3. Let $f(x) = e^x$.

- What is the Taylor series centered at zero for e^x?
- If we plan to use an n^{th} degree Taylor polynomial, $T_n(x)$, to approximate $f(x)$ for $-1 \leq x \leq 1$, what is an estimate for M?
- Find a value of n so that $T_n(x)$ approximates $f(x)$ with error less than 0.0001 for all x with $|x| \leq 1$.

e^1

$n = 7$

error $< \dfrac{e^1}{(n+1)!} 1^n = \dfrac{e^1}{(n+1)!} < \dfrac{1}{10000}$

$\dfrac{e^1}{n+1}$ derivative

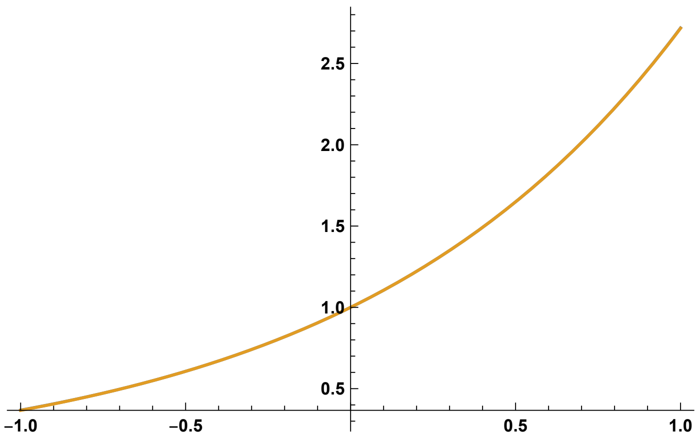
n = 7 (by trial or error)

In[320]:=
```
nterms = 5
f[x_] := Exp[x]
Series[f[x], {x, 0, nterms}]
sn[x_] = Normal[Series[f[x], {x, 0, nterms}]];
Plot[{f[x], sn[x]}, {x, -1, 1}]
Plot[{f[x] - sn[x]}, {x, -1, 1}, PlotLabel → "Absolute Error"]
```

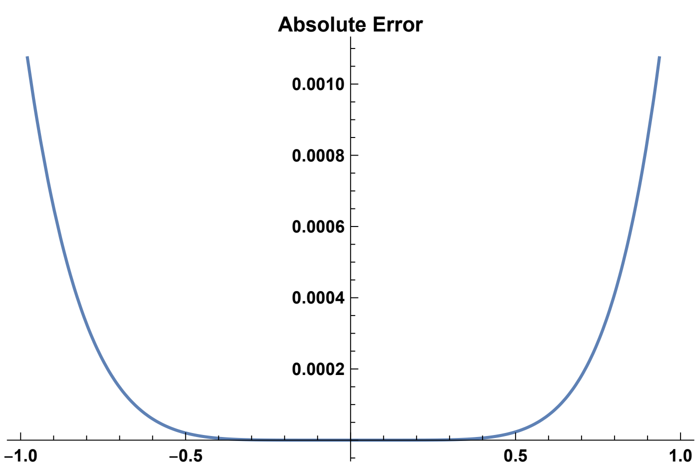
Out[320]= 5

Out[322]= $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O[x]^6$

Out[324]=



Out[325]=



(Video)