Center $x=a$

$$f(x) \simeq f(a)$$ (pretty bad)

$$\simeq f(a) + f'(a)(x-a)$$ (better!)

$$\simeq f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\vdots$$

$$\simeq f(a) + \dots + \underbrace{\frac{f^{(n)}(a)}{n!}}_{\text{coefficients}} \underbrace{(x-a)^n}_{\text{monomials}}$$

$$" = " \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$ (Taylor series)

best!

Example: $f(x) = \ln(1-x)$

$a = 0$

$$f(x) \simeq f(a) = f(0) = 0$$

$$f'(x) = \frac{-1}{1-x}$$

$$f^{(2)}(x) = \left[(-1)(1-x)^{-1}\right]'$$

$$= (-1)(1-x)^{-2}$$

$$f^{(3)}(x) = (-1)\,2\cdot(1-x)^{-3}$$

$$f^{(4)}(x) = (-1)\,3!\,(1-x)^{-4}$$

$\vdots$

$$f^{(n)}(x) = (-1)(n-1)!\,\boxed{(1-x)^{-n}}$$

$$f^{(n)}(0) = (-1)(n-1)!$$

$$f(x) \text{ "=" } \sum_{n=0}^{\infty} \underbrace{\frac{(-1)(n-1)!}{n!}}_{n \neq 0} (x-0)^n$$

$n = 0 \rightarrow f^{(0)}(0) = 0$

$$f(x) = \sum_{n=1}^{\infty} \frac{-1}{n} x^n = \boxed{(-1) \sum_{n=1}^{\infty} \frac{x^n}{n}}$$

$T_4(x)$ is the 4^{th} Taylor Polynomial

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

$$\underline{f'(x) = \frac{-1}{1-x}} = -\left(1 + x + x^2 + x^3 + \dots\right)$$

$$\left[-f(x)\right]' = \left[(-1)\sum_{n=1}^{\infty} \frac{x^n}{n}\right]'$$

$$= (-1) \sum_{n=1}^{\infty} \left(\frac{x^n}{n}\right)'$$

$$= (-1) \sum_{n=1}^{\infty} x^{n-1}$$

$$= \underbrace{-\left(1 + x + x^2 + x^3 + \dots\dots\right)}$$

Integrate term by term to get a series for $\underline{f}$.

Interval of convergence

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot x \right| = |x|$$

$\therefore$ convergence if $|x|<1$

$[-1, 1)$

What about endpoints?

$$f(x) \text{ " = " } (-) \sum_{n=1}^{\infty} \frac{x^n}{n} = \ln(1-x)$$

What error do we make in using $T_4(x)$ for $f(x)$?

$$R_4(x) = f(x) - T_4(x)$$

$$= (-1)\sum_{n=5}^{\infty} \frac{x^n}{n}$$

$$\left|R_n(x)\right| \leq \frac{M}{(n+1)!}\,|x-0|^{n+1}$$

M is an upper bound on the $(n+1)^{th}$ derivative

$$M \geq \left|f^{(n+1)}(x)\right| \text{ on the}$$

interval of interest around a.

$$f^{(5)}(x) = (-1)(5-1)! \cdot (1-x)^{-5}$$

If I want to use $T_4(x)$

on $[+.5, 1.5]$ ($x \in [-\frac{1}{2}, \frac{1}{2}]$)

$$|f^{(5)}(x)| \leq 4!/(\tfrac{1}{2})^5 = 32 \cdot 4! = M$$

$$|R_4(x)| \leq \frac{32 \cdot 4!}{5!} |x|^5 \quad \left(\begin{array}{c}\text{on} \\ x \in [-\frac{1}{2}, \frac{1}{2}]\end{array}\right)$$

$$\leq \frac{32}{5} \left(\frac{1}{2}\right)^5 = \frac{1}{5} \checkmark$$