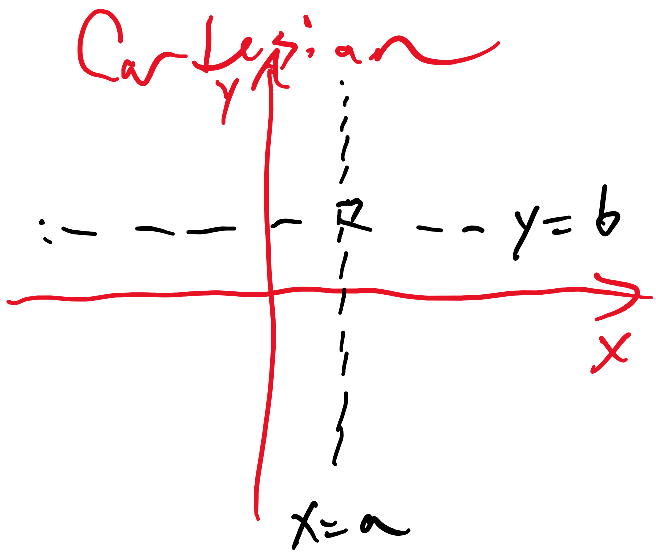
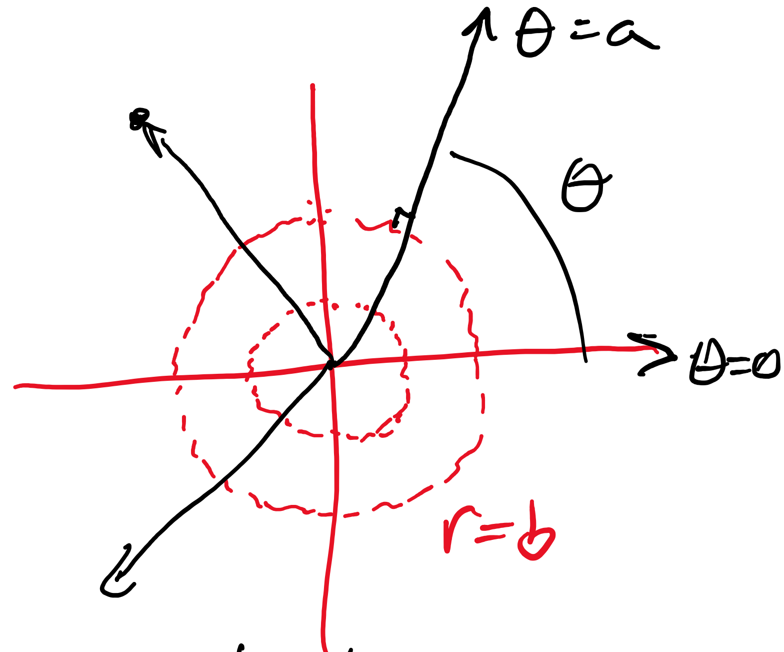
Lab 15: polar calculus





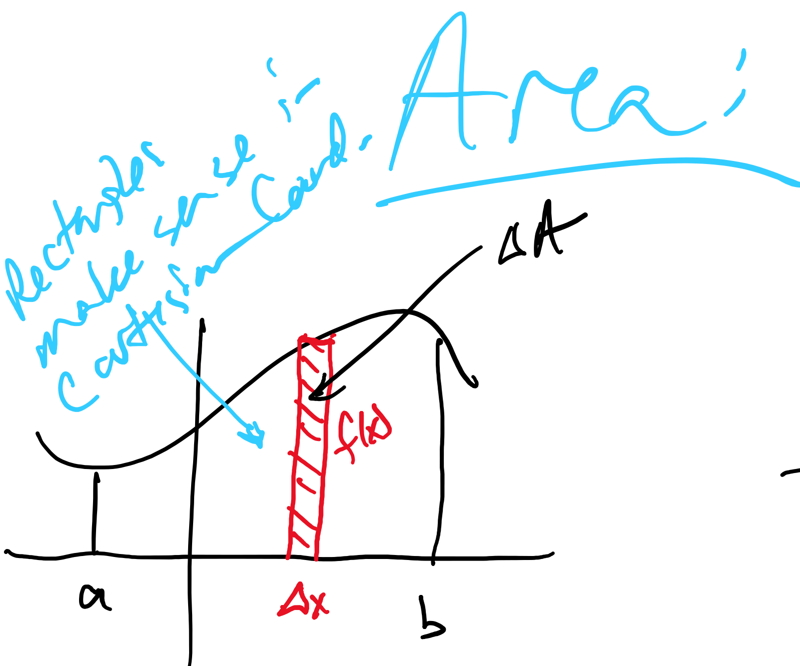
Might choose:

$r \geq 0$

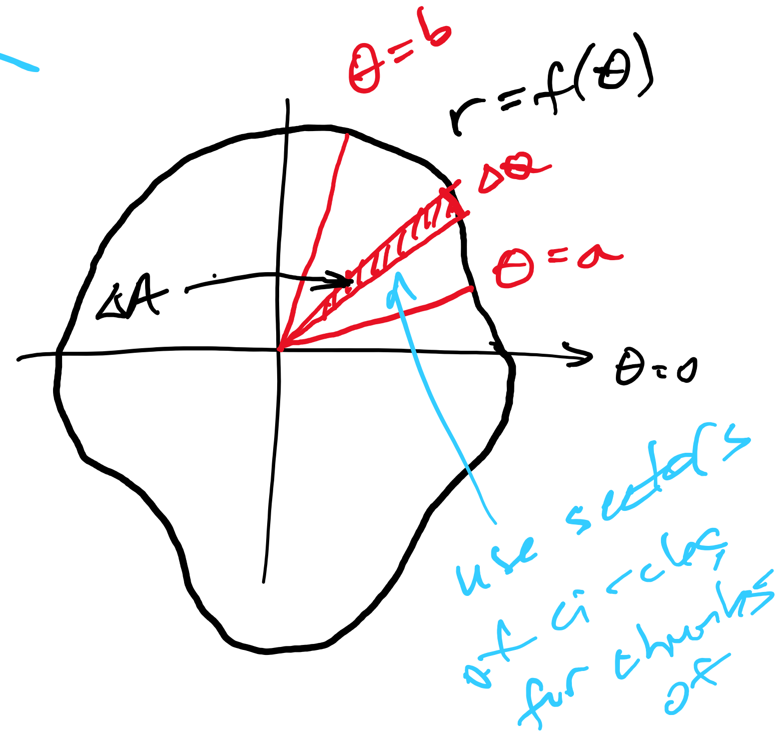
$\theta \in [0, 2\pi)$

(for <u>unique</u> representation)

Area:



$\Delta A = \Delta x \cdot f(x)$



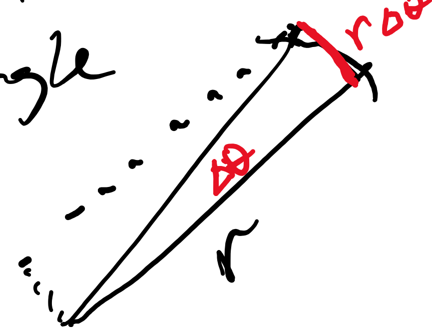
$\downarrow \; dA = dx \cdot f(x)$

$$A = \int_a^b dA = \int_a^b f(x)\,dx$$

$A = ?$

Area:

Each sector is like half a little rec-tangle

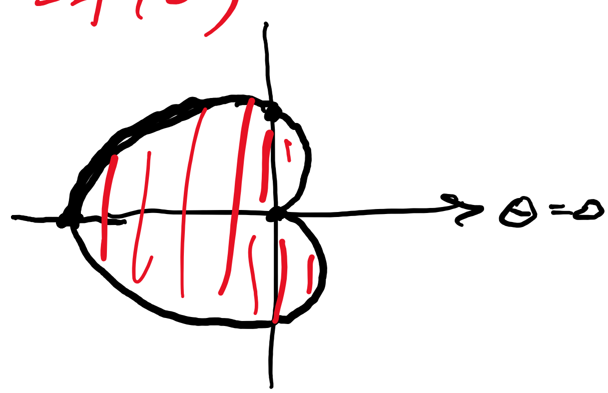


$$\Delta A \approx \frac{1}{2} L \cdot W = \frac{1}{2} r^2 \Delta\theta$$

$$dA = \frac{1}{2} r^2 d\theta$$

$$A = \int_a^b \frac{1}{2} r^2 d\theta = \int_a^b \frac{1}{2}\left(f(\theta)\right)^2 d\theta$$

Example: $r = 1 - \cos(\theta) = f(\theta)$



$$A = \frac{1}{2}\int_0^{2\pi} \left(1 - \cos(\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$$

0

$$= \frac{1}{2}\left[\int_0^{2\pi} 1\, d\theta + \int_0^{2\pi} \cos^2\theta\, d\theta\right]$$

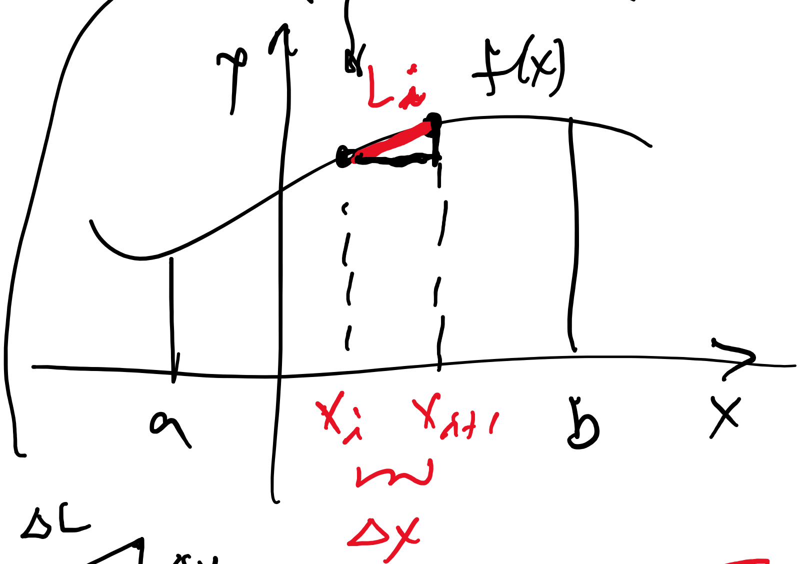
$$= \frac{1}{2}\left[2\pi + \int_0^{2\pi}\left(\frac{1}{2} + \underbrace{\cancel{\frac{1}{2}\cos 2\theta}}_{0}\right) d\theta\right]$$

Integral of sine or cosine over a period is 0

$$= \frac{1}{2}\left[2\pi + \pi\right] = \frac{3\pi}{2} \simeq 4.1$$

Length of a curve

Cartesian



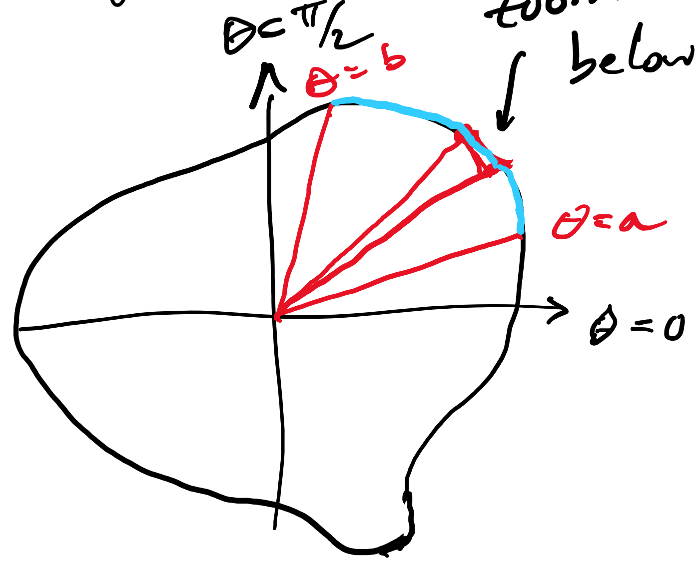
$$L_i = \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}$$

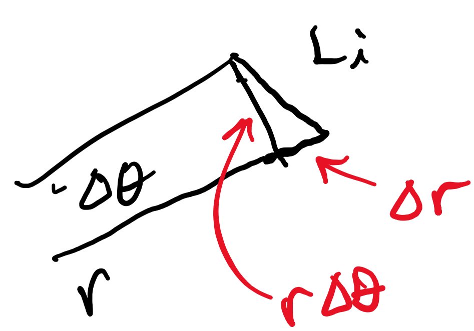
$$\Delta L = \sqrt{\Delta x^2 + \Delta y^2}$$

$$= \sqrt{\Delta x^2\left(1 + \frac{\Delta y^2}{\Delta x^2}\right)}$$

$$\Delta L = \Delta x\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

Polar





$$L_i^2 = \Delta r^2 + (r\Delta\theta)^2$$

$$dL = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\Delta L^2 = \Delta\theta^2\left[\frac{\Delta r^2}{\Delta\theta^2} + r^2\right]$$

$$\Delta L = \Delta\theta\sqrt{r^2+\left(\frac{\Delta r}{\Delta\theta}\right)^2}$$

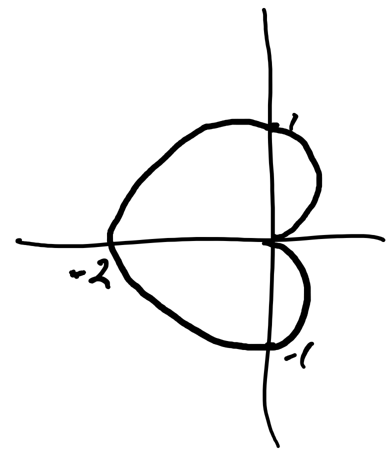
$$L = \int_a^b dL$$

$$\boxed{L = \int_a^b \sqrt{1+(f'(x))^2}\,dx}$$

$$dL = d\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$\boxed{L = \int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}$$

Example: $r = 1-\cos\theta$



$$L = \int_0^{2\pi}\sqrt{(1-\cos\theta)^2+(\sin(\theta))^2}\,d\theta$$

$$= \int_0^{2\pi}\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta$$

$$= \int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta$$

$$= \int_0^{2\pi}\sqrt{4\left(\tfrac{1}{2}-\tfrac{1}{2}\cos\theta\right)}\,d\theta$$

$$= \int_0^{2\pi} 2\sqrt{\sin^2\left(\tfrac{\theta}{2}\right)}\,d\theta = 2\int_0^{2\pi}\left|\sin\left(\tfrac{\theta}{2}\right)\right|d\theta$$

$$= 2\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta = 2\cdot(-2)\cdot \cos\left(\frac{\theta}{2}\right)\Big|_0^{2\pi}$$

$$= -4\left[-1 - 1\right] = \boxed{8}$$