Chapter 13: Vector Functions Kepler’s Laws of Planetary Motion
Book Title: Calculus
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Kepler’s Laws of Planetary Motion

We now describe one of the great accomplishments of calculus by showing how the material of this chapter can be used to prove Kepler’s laws of planetary motion. After 20 years of studying the astronomical observations of the Danish astronomer Tycho Brahe, the German mathematician and astronomer Johannes Kepler (1571–1630) formulated the following three laws.

Kepler’s Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

In his book *Principia Mathematica* of 1687, Sir Isaac Newton was able to show that these three laws are consequences of two of his own laws, the Second Law of Motion and the Law of Universal Gravitation. In what follows we prove Kepler’s First Law. The remaining laws are left as exercises (with hints).

Since the gravitational force of the sun on a planet is so much larger than the forces exerted by other celestial bodies, we can safely ignore all bodies in the universe except the sun and one planet revolving about it. We use a coordinate system with the sun at the origin and we let $\mathbf{r} = \mathbf{r}(t)$ be the position vector of the planet. (Equally well, $\mathbf{r}$ could be the position vector of the moon or a satellite moving around the earth or a comet moving around a star.) The velocity vector is $\mathbf{v} = \mathbf{r}'$ and the acceleration vector is $\mathbf{a} = \mathbf{r}''$. We use the following laws of Newton:

All models are wrong, some models are useful....

Second Law of Motion:

$$\mathbf{F} = m\mathbf{a}$$

Law of Gravitation:

$$\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{r^2}\mathbf{u}$$

Inverse square law...
Center-seeking force
(along the direction of
r, attached to the sun).

where $\mathbf{F}$ is the gravitational force on the planet, m and M are the masses of the planet and the sun, G is the gravitational constant, $r = |\mathbf{r}|$, and $\mathbf{u} = (1/r)\mathbf{r}$ is the unit vector in the direction of $\mathbf{r}$.

We first show that the planet moves in one plane. By equating the expressions for $\mathbf{F}$ in Newton's two laws, we find that

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

and so $\mathbf{a}$ is parallel to $\mathbf{r}$. It follows that $\mathbf{r} \times \mathbf{a} = 0$. We use Formula 5 in Theorem 13.2.3 to write

$$\begin{aligned} \tfrac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \mathbf{r}' \times \mathbf{v} + \mathbf{r} \times \mathbf{v}' \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{0} + \mathbf{0} = \mathbf{0} \end{aligned}$$

Therefore

$$\mathbf{r} \times \mathbf{v} = \mathbf{h}$$

where $\mathbf{h}$ is a constant vector. (We may assume that $\mathbf{h} \neq \mathbf{0}$; that is, $\mathbf{r}$ and $\mathbf{v}$ are not parallel.) This means that the vector $\mathbf{r} = \mathbf{r}(t)$ is perpendicular to $\mathbf{h}$ for all values of t, so the planet always lies in the plane through the origin perpendicular to $\mathbf{h}$. Thus the orbit of the planet is a plane curve.

To prove Kepler's First Law we rewrite the vector $\mathbf{h}$ as follows:

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{r}' = r\mathbf{u} \times (r\mathbf{u})' \\ &= r\mathbf{u} \times (r\mathbf{u}' + r'\mathbf{u}) = r^2(\mathbf{u} \times \mathbf{u}') + rr'(\mathbf{u} \times \mathbf{u}) \\ &= r^2(\mathbf{u} \times \mathbf{u}') \end{aligned}$$

Then

$$\begin{aligned} \mathbf{a} \times \mathbf{h} &= \frac{-GM}{r^2}\mathbf{u} \times (r^2\,\mathbf{u} \times \mathbf{u}') = -GM\,\mathbf{u} \times (\mathbf{u} \times \mathbf{u}') \\ &= -GM[(\mathbf{u} \cdot \mathbf{u}')\,\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u}'] \end{aligned}$$

(by Theorem 12.4.11, Property 6)

But $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}^2| = 1$ and, since $|\mathbf{u}(t)| = 1$, it follows from Example 13.2.4 that

$$\mathbf{u}\cdot\mathbf{u}' = 0$$

Therefore

$$\mathbf{a}\times\mathbf{h} = GM\,\mathbf{u}'$$

and so

$$(\mathbf{v}\times\mathbf{h})' = \mathbf{v}'\times\mathbf{h} = \mathbf{a}\times\mathbf{h} = GM\,\mathbf{u}'$$

Integrating both sides of this equation, we get

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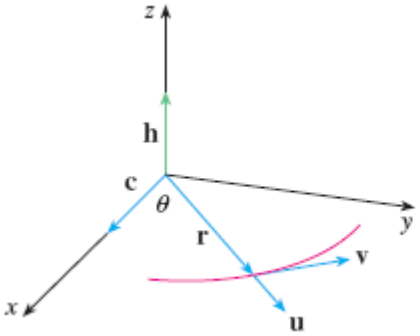
$$\mathbf{v}\times\mathbf{h} = GM\,\mathbf{u} + \mathbf{c}$$

where $\mathbf{c}$ is a constant vector.

At this point it is convenient to choose the coordinate axes so that the standard basis vector $\mathbf{k}$ points in the direction of the vector $\mathbf{h}$. Then the planet moves in the xy-plane. Since both $\mathbf{v}\times\mathbf{h}$ and $\mathbf{u}$ are perpendicular to $\mathbf{h}$, Equation 11 shows that $\mathbf{c}$ lies in the xy-plane. This means that we can choose the x-and y-axes so that the vector $\mathbf{i}$ lies in the direction of $\mathbf{c}$, as shown in Figure 8.

Figure 8



If θ is the angle between $\mathbf{c}$ and $\mathbf{r}$, then (r, θ) are polar coordinates of the planet. From Equation 11 we have

$$\begin{aligned}\mathbf{r}\cdot(\mathbf{v}\times\mathbf{h}) &= \mathbf{r}\cdot(GM\,\mathbf{u}+\mathbf{c}) = GM\,\mathbf{r}\cdot\mathbf{u}+\mathbf{r}\cdot\mathbf{c}\\ &= GMr\,\mathbf{u}\cdot\mathbf{u}+|\mathbf{r}||\mathbf{c}|\cos\theta = GMr + rc\cos\theta\end{aligned}$$

where $c = |\mathbf{c}|$. Then

$$r = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{GM + c \cos \theta} = \frac{1}{GM} \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e \cos \theta}$$

where $e = c/(GM)$. But

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = |\mathbf{h}|^2 = h^2$$

where $h = |\mathbf{h}|$. So

$$r = \frac{h^2/(GM)}{1 + e \cos \theta} = \frac{eh^2/c}{1 + e \cos \theta}$$

Writing $d = h^2/c$, we obtain the equation

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$$r = \frac{ed}{1 + e \cos \theta}$$

Comparing with Theorem 10.6.6, we see that Equation 12 is the polar equation of a conic section with focus at the origin and eccentricity e. We know that the orbit of a planet is a closed curve and so the conic must be an ellipse.

This completes the derivation of Kepler's First Law. We will guide you through the derivation of the Second and Third Laws in the Applied Project. The proofs of these three laws show that the methods of this chapter provide a powerful tool for describing some of the laws of nature.

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