Lab 6: Overview

Week 6, February 15

MAT 229, Spring 2021

Techniques of integration

- Integration by parts $\int u\,dv = u\,v - \int v\,du$
- Trigonometric integrals: $\int \sin^m(x)\cos^n(x)\,dx$, $\int \tan^m(x)\sec^n(x)\,dx$, $\int \cot^m(x)\csc^n(x)\,dx$
- Trigonometric substitutions
  - roots of $a^2 - x^2 \longrightarrow x = a\sin(\theta),\ dx = a\cos(\theta)\,d\theta$
  - roots of $a^2 + x^2 \longrightarrow x = a\tan(\theta),\ dx = a\sec^2(\theta)\,d\theta$
  - roots of $x^2 - a^2 \longrightarrow x = a\sec(\theta),\ dx = a\sec(\theta)\tan(\theta)\,d\theta$

Example

Evaluate the definite integral $\int_0^3 \frac{1}{\sqrt{16+x^2}}\,dx$

```
In[1]:= Integrate[1 / Sqrt[16 + x^2], {x, 0, 3}]
        N[%]
        Log[2.0]
        true = Log[2]

Out[1]= ArcSinh[3/4]

Out[2]= 0.693147

Out[3]= 0.693147

Out[4]= Log[2]
```

Numerical integration

Left endpoint rule -- L_n

To estimate $\int_a^b f(x)\,dx$ using n rectangles

- Width: $\Delta x = \frac{b-a}{n}$
- We subdivide the interval [a,b] into (n+1) x-values:
  - $x_0 = a + 0\,\Delta x = a$,
  - $x_1 = a + 1\,\Delta x$,
  - $x_2 = a + 2\,\Delta x$, ,...,
  - $x_k = a + k\,\Delta x$, ...,
  - $x_n = a + n\,\Delta x = a + n\,\frac{b-a}{n} = a + (b-a) = b$

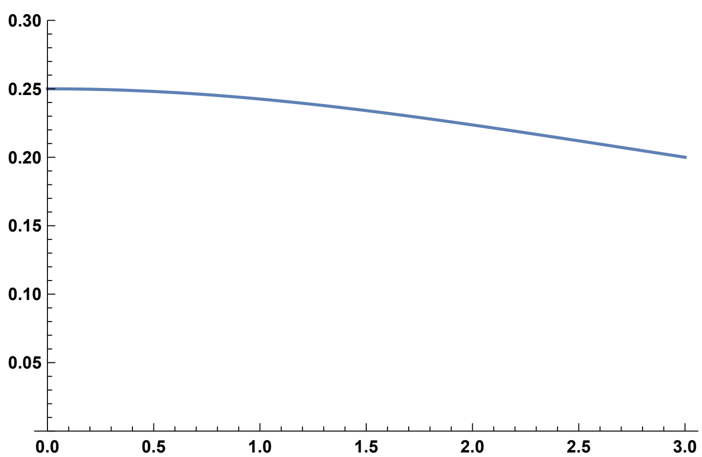
$$\int_a^b f(x)\,dx \approx \Delta x\left(\sum_{k=1}^{n} f(a + (k-1)\,\Delta x)\right)$$

Example

Estimate the definite integral $\int_0^3 \frac{1}{\sqrt{16+x^2}}\,dx$ using the left endpoint rule with $n = 50$.

In[5]:=
```
f[x_] := 1 / Sqrt[16 + x^2]
Plot[f[x], {x, 0, 3}, PlotRange → {0, 0.3}]
```

Out[6]=



In[17]:=
```
Clear[a, b, n, dx]
a = 0.0
b = 3.0
n = 50
dx = (b - a) / n
```

Out[18]= 0.

Out[19]= 3.

Out[20]= 50

Out[21]= 0.06

In[27]:=
```
lrr50 = dx * Sum[f[a + (k - 1) dx], {k, 1, 50}]
true = Log[2.0]
lrr50 - true
```
Out[27]= 0.69464

Out[28]= 0.693147

Out[29]= 0.0014928

Right endpoint rule -- R_n

To estimate $\int_a^b f(x)\,dx$ using n rectangles

- Same Width: $\Delta x = \frac{b-a}{n}$
- Same x-values: $x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\,\Delta x, x_3 = a + 3\,\Delta x, \ldots, x_k = a + k\,\Delta x, \ldots, x_n = a + n\,\Delta x = b$

$$\int_a^b f(x)\,dx \approx \Delta x \left(\sum_{k=1}^{n} f(a + k\,\Delta x)\right)$$

Example

Estimate the definite integral $\int_0^3 \frac{1}{\sqrt{16+x^2}}\,dx$ using the right endpoint rule with $n = 50$.

In[33]:=
```
rrr50 = Sum[dx * f[a + k dx], {k, 1, 50}]
true
rrr50 - true
```
Out[33]= 0.69164

Out[34]= 0.693147

Out[35]= -0.0015072

Midpoint rule

To estimate $\int_a^b f(x)\,dx$ using n rectangles

- Same Width: $\Delta x = \frac{b-a}{n}$
- **Different** x-values: $x_1 = a + \frac{1}{2}\Delta x, x_2 = a + \frac{3}{2}\Delta x, x_3 = a + \frac{5}{2}\Delta x, \ldots, x_k = a + \frac{2k-1}{2}\Delta x = a + \left(k - \frac{1}{2}\right)\Delta x, \ldots,$ $x_n = a + \left(n - \frac{1}{2}\right)\Delta x = a + (b - a) - \frac{1}{2}\Delta x = b - \frac{1}{2}\Delta x$

$$\int_a^b f(x)\,dx \approx \Delta x \left(\sum_{k=1}^{n} f\left(a + \left(k - \tfrac{1}{2}\right)\Delta x\right)\right)$$

Example

Estimate the definite integral $\int_0^3 \frac{1}{\sqrt{16+x^2}}\,dx$ using the midpoint rule with $n = 50$.

In[37]:=
```
mid50 = dx * Sum[f[a + (k - 1/2) dx], {k, 1, 50}]
true
mid50 - true
```

Out[37]= 0.693151

Out[38]= 0.693147

Out[39]= 3.60005×10^{-6}

Trapezoid rule

$\int_a^b f(x)\, dx \approx \frac{1}{2}(L_n + R_n)$

Example

Estimate the definite integral $\int_0^3 \frac{1}{\sqrt{16+x^2}}\, dx$ using the trapezoid rule with $n = 50$.

In[40]:=
```
trap50 = 1/2 (lrr50 + rrr50)
true
trap50 - true
```

Out[40]= 0.69314

Out[41]= 0.693147

Out[42]= -7.20006×10^{-6}

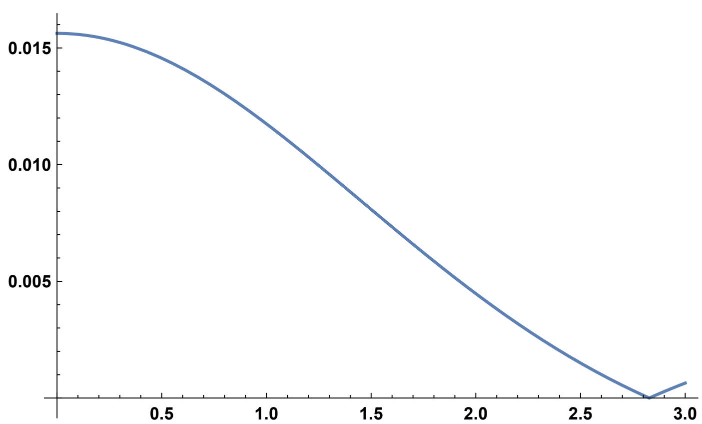
Error estimate

For the trapezoid rule the absolute error is less than or equal to

$\frac{K(b-a)^3}{12 n^2}$

In[43]:=
```
Plot[Abs[f''[x]], {x, a, b}]
```

Out[43]=



In[45]:=
```
K2 = Abs[f''[a]]
K2 = .016
```
Out[45]= 0.015625

Out[46]= 0.016

In[47]:=
```
trapErr = K2 (b - a) ^3 / (12 * n^2)
```
Out[47]= 0.0000144

In[48]:=
```
midErr = K2 (b - a) ^3 / (24 * n^2)
```
Out[48]= 7.2×10^{-6}

Simpson’s rule

To estimate $\int_a^b f(x)\,dx$ using $2n$ rectangles (always even)

- **Both sets of** *x*-values; those for Trapezoidal, and those for Midpoint
- Double the n for the corresponding Trapezoidal and Midpoint Rules.

$\int_a^b f(x)\,dx \approx S_{2n} = \frac{1}{3}(2 M_n + T_n)$

Example

Estimate the definite integral $\int_0^3 \frac{1}{\sqrt{16+x^2}}\,dx$ using Simpson’s rule with ***n* = 100**.

In[49]:=
```
S100 = (2 * mid50 + trap50) / 3
true
S100 - true
```
Out[49]= 0.693147

Out[50]= 0.693147

Out[51]= 1.55513×10^{-11}

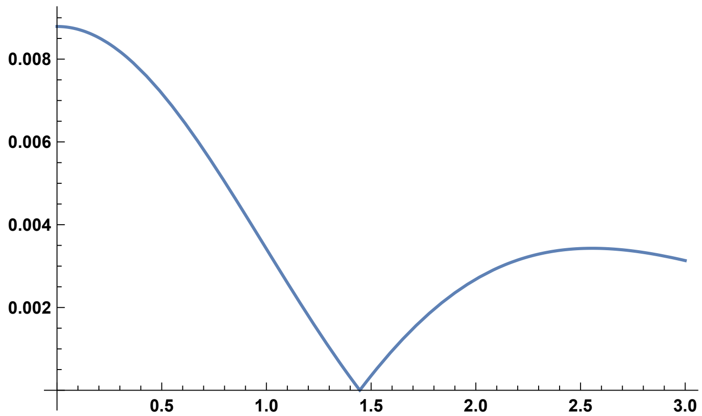
Error estimate

For Simpson’s rule the absolute error is less than or equal to

$\frac{K(b-a)^5}{180\,n^4}$

In[52]:=
```
Plot[Abs[f''''[x]], {x, 0, 3}]
```

Out[52]=



In[54]:=
```
K4 = Abs[f''''[0.]]
K4 = 0.009
```

Out[54]= 0.00878906

Out[55]= 0.009

In[56]:=
```
K4 (b - a) ^5 / (180 * (2 n) ^4)
```

Out[56]= 1.215×10^{-10}