Huygens' Cycloidal Clock

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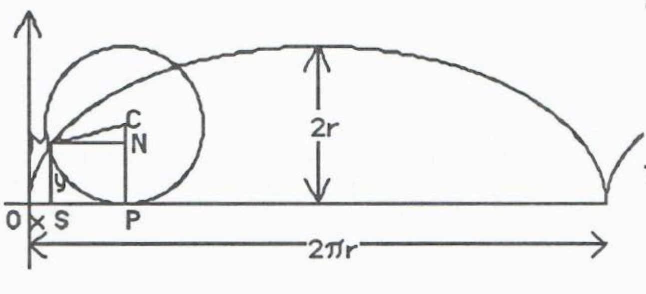
Galileo conjectured the (near-) isochronous property of the pendulum but relied on water clocks to measure time. The first pendulum clock was invented in 1656 by the great mathematician and physicist Huygens. Not satisfied with the accuracy of the pendulum clock, Huygens sought to devise a pendulum with period strictly independent of the amplitude. The path of the bob of such a pendulum is called a tautochrone. Huygens proved that the cycloid is a tautochrone, and that its evolute is a congruent cycloid. These insights provided the theoretical basis for clocks (built around 1700) in which cycloidal jaws forced the bob to move along a cycloidal path.

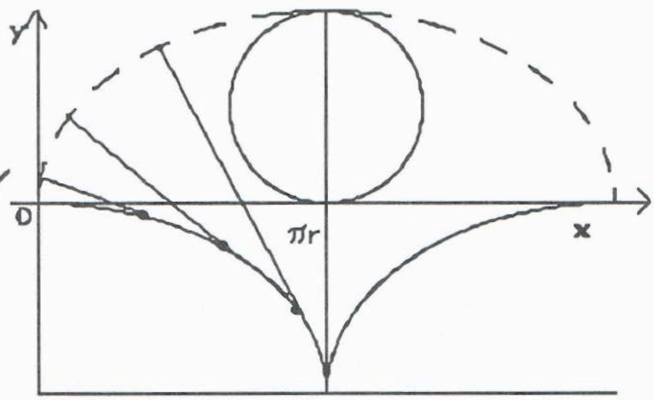
While the practical value of Huygens' insights proved insignificant (pendulum friction and air resistance overshadow the improvement resulting from replacing an ordinary pendulum with a cycloidal pendulum), his reasoning in the proof that the cycloid is the *only* tautochrone "goes far beyond the differential and integral calculus . . .

What follows are three computations, marked (a), (b), and (c), respectively. (a) is a calculus-based proof of the tautochrone property of the cycloid. (b) is Huygens' proof of this property. (c) is an application of the setup in (b).

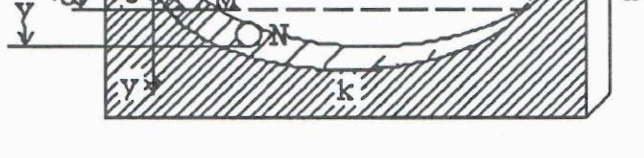
Huygens' proof (b) is brilliant but beyond the inventive capacity of ordinary mortals. The calculus proof (a) calls for a measure of skill in the handling of integrals but is essentially routine. A comparison of the two



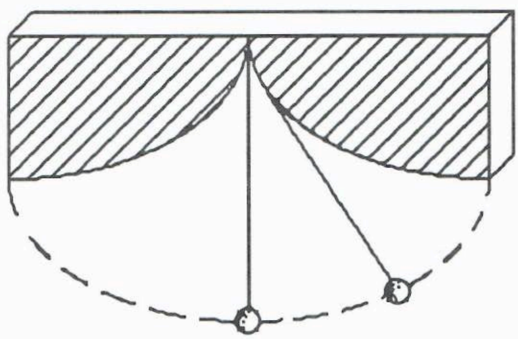
Genesis of the cycloid



The evolute of a cycloid is a cycloid



The cycloidal clock "in the raw"



The cycloidal clock "in finished form"

solutions is a dramatic illustration of the "routinizing" power of the calculus.

(a) *How to show that a cycloid is a tautochrone by the use of calculus*

For uniform motion (that is, motion with constant velocity), the time of travel t, the distance travelled s, and the velocity v are related by

$$v = \frac{s}{t}$$

Hence

$$t = \frac{s}{v}$$

The integral calculus generalizes this to nonuniform motion. If the velocity v depends on the distance s travelled, then the formula for the time t of travel is

(l) $$t = \int \frac{ds}{v},$$

the limits of integration being the initial and final distances, respectively.

Suppose the motion is along a curve given parametrically as

$$x = x(u)\ ,\ y = y(u)$$

Then

$$ds = \sqrt{x'^2 + y'^2}\, du$$

where ' denotes differentiation with respect to the parameter. Setting this into the integral (l) for the time of travel gives

(l)' $$t = \int \frac{\sqrt{x'^2 + y'^2}}{v}\, du .$$

To evaluate this integral we need to know v as a function of the parameter. To do that we apply the law of conservation of energy. If a unit mass particle is at rest at the point

$$M = (x_o, y_o) = (\, x(u_o)\, ,\, y(u_o)\,)$$

(see Figure 1),

then its kinetic energy $\frac{1}{2}v^2$ at the point $N = (x,y)$ is equal to the loss of potential energy between M and N. This loss is $g(y - y_o)$, so

(2) $$\frac{1}{2}v^2(u) = g(y-y_o)$$

Now take for the curve a cycloid, given parametrically by

$$x = r\,(u - \sin u), \qquad y = r\,(1 - \cos u).$$

Then

$$x' = r\,(1 - \cos u), \quad y' = r \sin u .$$

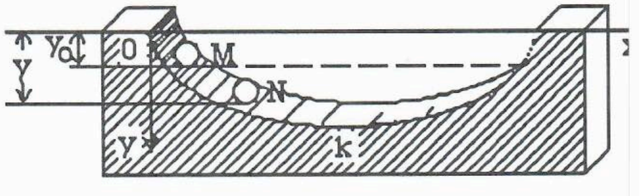


Figure 1

Hence

$$\sqrt{x'^2 + y'^2} = r\sqrt{2 - 2\cos u}\ ,$$

and, from (2),

$$v = \sqrt{2gr(\cos u_o - \cos u)}\ .$$

Setting these into (l)' gives for the time of travel between M and N

$$t = \sqrt{\frac{r}{g}} \int_{u_o} \sqrt{\frac{1 - \cos u}{\cos u_o - \cos u}}\, du$$

Using the identities

$$1 - \cos u = 2 \sin^2 \frac{u}{2},$$

$$\cos u = 2 \cos^2 u - 1\ ,$$

we can rewrite the above as

$$t = \sqrt{\frac{r}{g}} \int_{u_o} \frac{\sin \frac{u}{2}}{\sqrt{\cos^2 \frac{u_o}{2} - \cos^2 \frac{u}{2}}}\, du .$$

Now take N to be the point at the bottom of the cycloid. This corresponds to the parameter value $u = \pi$, so the time taken to reach the bottom starting at M is

$$t = \sqrt{\frac{r}{g}} \int_{u_o}^{\pi} \frac{\sin \frac{u}{2}}{\sqrt{\cos^2 \frac{u_o}{2} - \cos^2 \frac{u}{2}}}\, du .$$

Introduce $\cos\frac{u_o}{2}$ as a new variable of integration. Then the limits are $p_o = \cos\frac{u_o}{2}$ and $\cos\frac{\pi}{2} = 0$, so the time of travel is given by the integral

$$t = 2\sqrt{\frac{r}{g}} \int_0^{p_o} \frac{dp}{\sqrt{p_o^2 - p^2}} = \pi \sqrt{\frac{r}{g}}$$

This is a remarkable result! It shows that the time it takes for the particle to get from the release point M to the lowest point K on the cycloid is independent of M, that is, the cycloid is a tautochrone. Q.E.D.

(b) *How Huygens showed (more than 300 Years ago) that the cycloid is a tautochrone*

If two particles are released from the same height and move along two curves with velocities whose *vertical* components are the same, then they will reach ground level at the same time.

Huygens' idea is to replace the gravity-induced motion of a particle on a cycloid with a motion of a particle on a circle such that the vertical components of the velocities of the two particles are the same. The details follow.

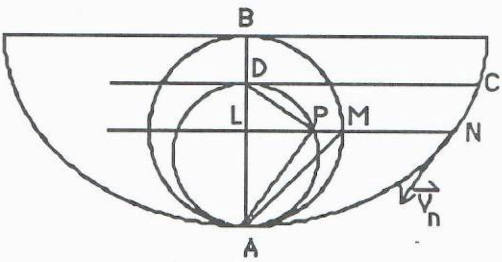


Figure 2

Consider the cycloid generated by the circle with diameter AB (Figure 2). The particle on the cycloid is released at C (Figure 2). N is a "typical" position of this particle. The corresponding particle moves on the semicircle DPA with diameter DA, where D is at the same height as C and A is the lowest point on the cycloid. P is a "typical" position of that particle. We recall that our aim is to impart to the particle on the semicircle a velocity whose vertical component is equal to the vertical component of the gravity-induced velocity of the particle on the cycloid.

We have: $|\vec{v}_n| = \sqrt{2g\,DL}$. A (nonobvious!) geometric fact of fundamental importance is that *the tangent to the cycloid at N is parallel to the chord AM* (this can be easily deduced by calculus from the parametric equations of the cycloid). This being so, the vertical component of $\vec{v}_n$ is

$\sqrt{2g \cdot DL} \cdot \sin\angle AML$, that is, $\sqrt{2g \cdot DL}\ \dfrac{AL}{AM}$.

Now we replace AM with quantities in the semicircle DPA. Since $LP^2 = DL \cdot AL$ implies that $LP = \sqrt{DL \cdot AL}$, and $AM^2 = AL \cdot AB$ implies that $AM = \sqrt{AL} \cdot \sqrt{AB}$, it follows that the magnitude of the vertical component of $\vec{v}_n$ is

$$\sqrt{2g\,DL}\,\frac{AL}{AM} = \sqrt{2g}\ \frac{\sqrt{DL \cdot AL} \cdot \sqrt{AL}}{\sqrt{AL} \cdot \sqrt{AB}}$$

$$= \sqrt{\frac{2g}{AB}}\ LP\ .$$

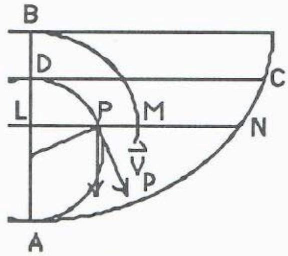


Figure 3

The vertical component of the (as yet undetermined) velocity $\vec{v}_p$ is $|\vec{v}_p| \cdot LP/\frac{AD}{2}$ (see Figure 3). Since we insist on the equality $2|\vec{v}_p| \cdot \frac{LP}{AD} = \sqrt{\frac{2g}{AB}}\ LP$, it follows that $|\vec{v}_p| = \frac{AD}{2}\sqrt{2g/AB}$. But then the time it takes the particle moving on the semicircle DPA to get from D to A is

$$\pi \cdot \frac{\frac{AD}{2}}{\sqrt{2g/AB} \cdot \frac{AD}{2}} = \pi / \sqrt{2g/AB}\ ,$$

which is independent of the elevation of D. It follows that *the time of descent of the particle moving under the action of gravity from C to A along the cycloidal arc CNA is independent of the elevation of C*, that is, that *the cycloid is a tautochrone* and its amplitude-independent quarter-period is $\pi/\sqrt{2g/AB}$. (N.B. Putting AB = 2r , where r is the radius of the circle generating the cycloid in our first computation, we see that the quarter-period value obtained by Huygens is also $\pi\sqrt{r/g}$.

Historical note. Since π was introduced in the 18th century (by Euler) it could not have been used by Huygens. Huygens' result was that

$$\frac{\text{CNA-time}}{\text{free-fall BA-time}} = \frac{\text{DPA-time}}{\text{free-fall BA-time}}$$

$$= \frac{\text{length of DPA}}{|\vec{v}_p|\ \sqrt{2AB/g}}$$

$$= \frac{\text{length of DPA}}{\frac{AD}{2}\sqrt{2g/AB}\ \sqrt{2AB/g}}$$

$$= \frac{\text{length of DPA}}{AD}\ .$$

This shows that the CNA-time, that is the time of descent of the particle on the cycloid from C to A , is independent of C. Q.E.D.

(c) *An application of the setup in (b)*

Let us introduce yet another particle into the Huygens scheme. This particle will be the projection—one might say the shadow—on AD of the particle traversing the semicircle DPA. The reason for our interest in this particle is that its velocity is directed vertically and equal to the vertical components of the velocities of the two previous particles. Since the speed of the particle on the semicircle DPA is $\frac{AD}{2}\sqrt{2g/AB} = \frac{AD}{2}\sqrt{g/r}$, its angular velocity is $\omega = \sqrt{g/r}$. As we vary the release point C on the cycloid we obtain a family of semicircles DPA traversed with angular velocity ω. To each constant-speed motion on a semicircle of the family there corresponds a "shadow-motion" along the corresponding diameter DA (see Figure 2). When the shadow particle is at L its speed is $\sqrt{g/r} \cdot LP = \omega LP$. The shadow is executing a simple harmonic motion with half-period π/ω. If we introduce a coordinate system with center at the midpoint of DA and x-axis determined by AD, then the half-period of the shadow-motion is

$$\int_{-\rho}^{\rho}\frac{dx}{\omega LP} = \frac{1}{\omega}\int_{-\rho}^{\rho}\frac{dx}{\sqrt{\rho^2 - x^2}}$$

, where ρ = AD/2. The equality $\frac{1}{\omega}\int_{-\rho}^{\rho}\frac{dx}{\sqrt{\rho^2 - x^2}} = \frac{\pi}{\omega}$ implies that $\int_{-\rho}^{\rho}\frac{dx}{\sqrt{\rho^2 - x^2}} = \pi$, regardless of the (positive) value of ρ. We have evaluated a nontrivial integral by giving it a physical interpretation!

While it is true that

$$\int_{-\rho}^{\rho}\frac{dx}{\sqrt{\rho^2 - x^2}} = \left[\arc\sin\frac{x}{\rho}\right]_{-\rho}^{\rho} = \pi ,$$

it is nice to be able to predict this by interpreting $\frac{1}{\omega}\int_{-\rho}^{\rho}\frac{dx}{\sqrt{\rho^2 - x^2}}$ as the (amplitude-independent!) half-period of a simple harmonic motion with frequency $\frac{\omega}{2\pi}$.

References

The computations in (a), originally based on 11, pp. 240-245 of A. A. Savelov, *Plane Curves*, Moscow, 1960 (in Russian), were made more explicit and transparent by Peter Lax.

(b) is adapted from Vol. 2, pp. 206-210 of *History of Mathematics*, ed. A. P. Yushkevich, Nauka, 1970 (in Russian).