Parametric Equations

MAT 229, Spring 2021

Week 13

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  Section 10.1: Parametric equations
- Boelkins/Austin/Schlicker's *Active Multivariable Calculus*
  Section 9.6: Vector-valued functions

Curves defined by parametric equations

Example

The location for Tara the Tyrannosaurus Rex at time t hours is given by

$$(x, y) = (\sin(t) - \sin(2\,t), \cos(t))$$

Time t is the parameter, and once it is known, the dinosaur's position (x, y) is known.

Questions

- What is Tara's location at time $t = 0$?
- What is Tara's location at time $t = \pi/2$?
- When does Tara return to the location she was at when t was 0?

(Video)

Definition

Parametric equations are functional values for x and y coordinates

$x = f(t)$

$y = g(t)$

usually presented as an ordered pair of coordinates in the plane:

(x,y) = (f(t),g(t))

Think of this as a curve (sometimes called a "space curve") in the plane, which is a function of *t* (generally thought of as time). This is an excellent way to represent a **motion**. And we can generalize this to (x,y,z) -- a curve in three-dimensional space.

Like if an apple falls on your head, or something like that -- the motion of the apple can be captured in a space curve, given by parametric equations for the motion of the apple.

Questions

Consider the parametric equations $x = \cos(t)$, $y = \sin(t)$.

- Make a table of values for these parametric equations for $t = 0, \pi/6, \pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4, 5\pi/6, \pi$. Then connect these points.
- Using the Pythagorean identity, find an equation on x and y these parametric equations satisfy.
- What curve do these parametric equations represent?

(Video)

Definition

A **parametric curve** is the set of all points represented by parametric equations $x = f(t)$, $y = g(t)$ for values of t over the given domain.

Graphing calculators and software like Mathematica can draw curves represented by given parametric equations.

Example

To draw the curve in Mathematica given by the parametric equations

$$x = \sin(t) + \frac{1}{2}\cos(5\,t) + \frac{1}{4}\sin(13\,t)$$

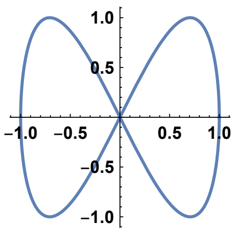
$$y = \cos(t) + \frac{1}{2}\sin(5\,t) + \frac{1}{4}\cos(13\,t)$$

for $0 \le t \le 2\pi$ enter

```
ParametricPlot[
 {Sin[t] + 1/2 Cos[5 t] + 1 / 4 Sin[13 t], Cos[t] + 1/2 Sin[5 t] + 1 / 4 Cos[13 t]},
 {t, 0, 2 Pi}
]
```

It’s kind of nice to watch the curve get traced out, which we can do with a “Manipulate” command in Mathematica.

Example: What’s the equation of Infinity?



Identifying curves defined by parametric equations

One may be able to identify a curve (like that infinity curve above) defined by parametric equations, by finding an equation in only x and y. Alternatively, you may want to create a motion that follows a given curve, given by x and y (e.g. you may want a robot in a bakery to trace a given curve, say laying down icing on a cake to wish someone a “Happy Birthday”).

Techniques

For the parametric equations $x = f(t)$, $y = g(t)$, two techniques for finding an equation on x and y:

- Find an identity between $f(t)$ and $g(t)$.
- Solve $x = f(t)$ -- $t=f^{-1}(x)$ -- and plug this value of t into $y = g(t) = g(f^{-1}(x))$; or *vice versa*.

Questions

- For parametric equations $x = 2\cos(t)$, $y = 2\sin(t)$, use the Pythagorean identity, $\cos^2(t) + \sin^2(t) = 1$. What do you get? (Video)
- For parametric equations $x = 2t$, $y = 1 + t^2$, solve $x = 2t$ for t. Then relate x to y by replacing t in the equation for y. What do you get? (Video)

Questions

Identify the curves defined by the parametric equations.

- $x = 2t + 1$, $y = 3t - 4$. (Video)
- $x = \cos(t) - 5$, $y = \sin(t) + 6$. (Video)
- $x = 2\cos(t)$, $y = 3\sin(t)$. (Video)

Homework

- IMath problems on Parametric Equations.