Cross Product

MAT 229, Spring 2021

Week 16

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  Section 12.4: Cross product
- Boelkins/Austin/Schlicker's *Active Multivariable Calculus*
  Section 9.4: The cross product

Review

Questions

- What are the components of the vector that points from location $(1, 0, -2)$ to location $(2, 4, 3)$?
- What is a unit vector that points in that same direction?

(Video)

Questions

- What is the geometric significance of scalar multiplication? (Video)
- What is the geometric significance of vector addition? (Video)
- What is the geometric significance of vector subtraction? (Video)
- What is the geometric significance of the dot product? (Video)

Cross product definition

The *dot product* is a multiplication-like operation between two vectors (two 2D vectors or two 3D vectors -- and beyond!) that gives a scalar value.

The *cross product* is a multiplication-like operation between two 3D that gives a vector value. **The cross product is only defined for 3D vectors.**

Geometric definition

The cross product of 3D vectors $\vec{u}$ and $\vec{v}$ is denoted $\vec{u} \times \vec{v}$.

- The magnitude of $\vec{u} \times \vec{v}$ is given by

$$\left| \vec{u} \times \vec{v} \right| = \left| \vec{u} \right| \left| \vec{v} \right| \sin(\theta)$$

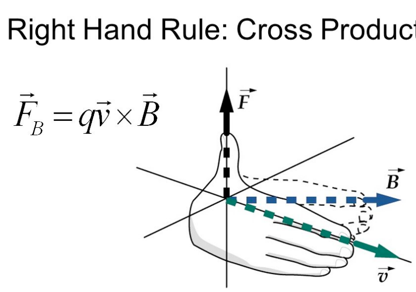
  where θ is the smaller of the two angles between vectors $\vec{u}$ and $\vec{v}$.

- The direction of $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. There are two such directions. Choose the one that satisfies the right-hand rule.

Right-hand rule

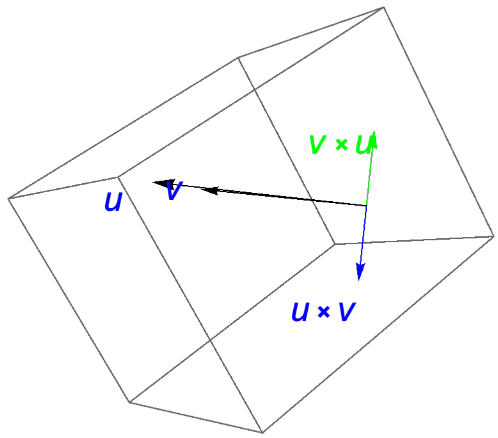
To determine the direction of $\vec{u} \times \vec{v}$ from the two possible directions, using the fingers on your right hand

- Point your index finger in the direction of $\vec{u}$.
- Sweep that finger towards the direction of $\vec{v}$. Your thumb will point in the correct direction of $\vec{u} \times \vec{v}$.



(source)

Magnetic force is a cross product. Torque is a cross product. There are lots of cross products in physics!



Questions

Let $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, and $\vec{k} = \langle 0, 0, 1 \rangle$. Use the above geometric definition to determine the following cross products.

- What is $\vec{i} \times \vec{j}$?
- What is $\vec{i} \times \vec{k}$?
- What is $\vec{j} \times \vec{k}$?

(Video)

Question

Are $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ the same? (Video)

Component definition

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then

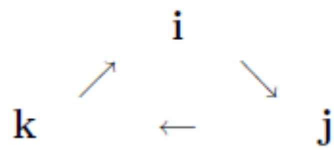
$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle.$$

Notes:

- The first component of the cross product consists of the second and third components of the given two vectors.
- The second component of the cross product consists of the first and third components of the given two vectors.
- The third component of the cross product consists of the first and second components of the given two vectors.

- If we check the **permutations** of the indices, then 123, 231, and 312 give rise to positive contributions, while 321, 213, and 132 give rise to negative contributions.

Since I hate memorizing stuff like all that above, I just think in terms of components, and the three basis vectors of our Cartesian coordinate system:



$$\vec{u} \times \vec{v} = \left(u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}\right) \times \left(v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}\right) \; ?$$
$$= (u_2 v_3 - u_3 v_2)\, \vec{i} + (u_3 v_1 - u_1 v_3)\, \vec{j} + (u_1 v_2 - u_2 v_1)\, \vec{k}$$

Questions

Let $\vec{i} = \langle 1, 0, 0\rangle$, $\vec{j} = \langle 0, 1, 0\rangle$, and $\vec{k} = \langle 0, 0, 1\rangle$. Use the above component definition to determine the following cross products. (Compare with the results using the geometric definition.)

- What is $\vec{i} \times \vec{j}$?
- What is $\vec{i} \times \vec{k}$?
- What is $\vec{j} \times \vec{k}$?

Question

What is a vector that is perpendicular to both $\vec{u} = \langle 2, -1, 3\rangle$ and $\vec{v} = \langle 1, 2, 1\rangle$? (Video)

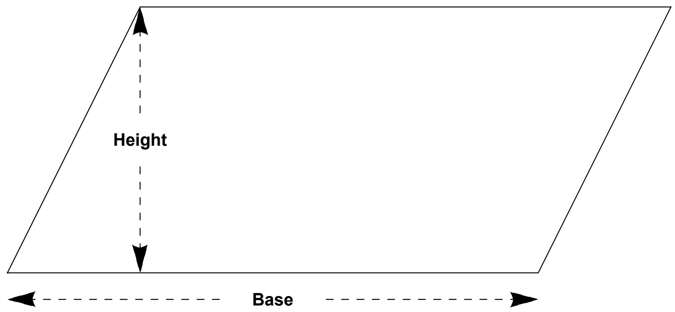
Cross product geometric properties

The magnitude of $\vec{u} \times \vec{v}$ is given by

$$\left|\vec{u} \times \vec{v}\right| = \left|\vec{u}\right| \left|\vec{v}\right| \sin(\theta)$$

where θ is the smaller of the two angles between vectors $\vec{u}$ and $\vec{v}$.

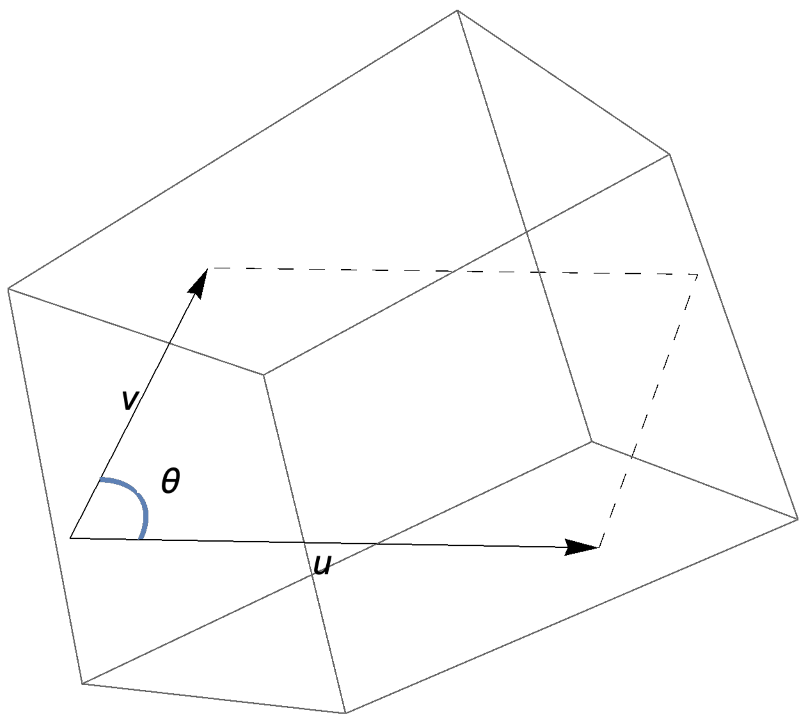
- If $\vec{u} \times \vec{v} = \vec{0}$, then its magnitude is 0. One of three things must be true.
  - $\vec{u} = \vec{0}$, or
  - $\vec{v} = \vec{0}$, or
  - $\sin(\theta) = 0$ which means $\theta = 0$. The vectors must be parallel, pointing in the same direction or in the exact opposite direction.
- Place vectors $\vec{u}$ and $\vec{v}$ so their initial points are the same. They form two of the sides of a parallelogram. The area of a parallelogram is height × base.



The area of this parallelogram is

$$\left|\vec{u}\right| \left|\vec{v}\right| \sin(\theta) = \left|\vec{u} \times \vec{v}\right|$$

(in terms of the lengths of these vectors and the angle θ between them).



Questions

- What is the area of the **parallelogram** formed from vectors $\vec{u} = \langle 2, -1, 3\rangle$ and $\vec{v} = \langle 1, 2, 1\rangle$? (Video)
- What is the area of the **triangle** with vertices $(0, 1, 0)$, $(2, -1, -1)$, $(-1, 0, 1)$? (Video)

Cross product algebraic properties

Question

How are $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ related? (The cross-product is non-**commutative!**)

Other properties

1. $\left(s\,\vec{a}\right)\times\vec{b}=s\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\left(s\,\vec{b}\right)$ (where s is a scalar)

2. $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

3. $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$

Questions

- Show property 2 is true by computing each side using
  $$\vec{a}=\langle a_1,\,a_2,\,a_3\rangle,\ \vec{b}=\langle b_1,\,b_2,\,b_3\rangle,\ \vec{c}=\langle c_1,\,c_2,\,c_3\rangle$$
  (Video)
- Show that the cross-product is non-**associative**: that is, in general
  $$\vec{a}\times\left(\vec{b}\times\vec{c}\right)\neq\left(\vec{a}\times\vec{b}\right)\times\vec{c}.$$
  (Video)
- What is $\vec{k}\times\left(\vec{i}+2\,\vec{j}\right)$?
  (Video)
- What is $\vec{a}\times\vec{b}=\left(a_1\,\vec{i}+a_2\,\vec{j}+a_3\,\vec{k}\right)\times\left(b_1\,\vec{i}+b_2\,\vec{j}+b_3\,\vec{k}\right)$?
  (Video)

Homework

- IMath problems on cross products.