Integration By Parts

MAT 229, Spring 2021

Week 4

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  Section 7.1: Integration by Parts
- Boelkins/Austin/Schlicker's Active Calculus
  Section 5.4: Integration by parts

Integration techniques

Anti-differentiation is the reverse of differentiation. Anything we can learn about anti-differentiation must come from differentiation. They're reflections of each other in some weird Alice-in-Wonderland looking-glass.

- Power rule and Power rule: $\frac{d}{dx}(x^n) = n\,x^{n-1} \longrightarrow \int x^k\,dx = \frac{1}{k+1}x^{k+1} + C$ if $k \neq -1$.
- Chain rule and Substitution: $\frac{d}{dx}(F(u(x))) = F'(u(x))\,u'(x) \longrightarrow \int f(u(x))\,u'(x)\,dx = \int f(u)\,du$

What about the Product rule? What's its mirror reflection?

We want an integration technique that comes from the product rule for differentiation.

$$(u(x)\,v(x))' = u'(x)\,v(x) + u(x)\,v'(x)$$

If the two sides above are equal as functions, so are their integrals: so integrating both sides produces

$$u(x)\,v(x) = \int u'(x)\,v(x)\,dx + \int u(x)\,v'(x)\,dx$$

Rewrite this as

$$u(x)\,v(x) - \int u'(x)\,v(x)\,dx = \int u(x)\,v'(x)\,dx$$

And there's your new rule, with the mysterious name of "Integration by Parts" (rather than "the product rule backwards").

Integration by parts

Here's how I usually come at it (using functions f and g, which are my favorites for the project rule). Let's say you don't like the integral $\int f(x)\, g'(x)\, dx$. You can't think of an anti-derivative of $f(x)\, g'(x)$. So rewrite it as

$$f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

and maybe that one will look better to you (replace one pesky integral for another): you might recognize the integrand $f'(x)\, g(x)$, and know an anti-derivative.

Note that this works for definite integrals, too: we simply add limits:

$$\int_a^b f(x)\, g'(x)\, dx = f(x)\, g(x) \Big|_a^b - \int_a^b f'(x)\, g(x)\, dx$$

So you've got choices. If one looks better, do that one!

Here's the key: look for a **product** in the integrand**,** of two functions: one you wouldn't mind differentiating (*f(x)*), and the other you wouldn't mind **anti**-differentiating (think of it as *g'(x)*). And maybe you'll get a product in the integrand that you like better.

Integration by parts -- alternate version

We should mention that some folks write integration by parts in a slightly different way: If you can identify $u(x)$ and $dv(x)$ in an integral, try rewriting the integral as

$$\int u(x)\, \mathrm{dv}(x) = u(x)\, v(x) - \int v(x)\, \mathrm{du}(x)$$

This comes right out of the forms above, since we can think of the product

$$v'(x)\mathrm{dx} = \frac{dv}{dx} dx = dv(x)$$

so

$$\int u(x)\, v'(x)\, \mathrm{dx} = \int u(x)\, \mathrm{dv}(x) = u(x)\, v(x) - \int v(x)\, \mathrm{du}(x)$$

It's really just a shorthand for what we've got above (and which I prefer).

People often suppress the x dependence: if you start with my favorite form, and if we write *u=f(x)* and *dv=g'(x)dx*, then *du=f'(x)dx* and *v=g(x);* and if we can identify an integral as

$$\int u\, dv = u\, v - \int v\, du.$$

This makes it all look a little like a double substitution. It's actually just a good shorthand. I personally prefer the first form we considered, above -- but you're welcome to use this alternative form (and sometimes I do!).

Example

To evaluate $\int x\, e^x\, dx$ let's try integration by parts. Identify $u(x)$ and $v'(x)$.

$$u(x) = x$$

$$v'(x) = e^x$$

This leads to

$$u'(x) = 1$$

$$v(x) = \int v'(x)\, dx = \int e^x\, dx = e^x$$

Only one antiderivative is needed; any value of C at this point in "+C" works. Applying the integration by parts formula $\int u\,dv = u\,v - \int v\,du$ produces

$$\int x\,e^x\,dx = x\,e^x - \int e^x\,dx$$

The new integral is one of our basic integrals. The final answer is

$$\int x\,e^x\,dx = x\,e^x - e^x + C$$

Alternate form: Choose $u = x$ and $dv = e^x\ dx$; then $du = dx$ and $v = e^x$, so that the process looks like this:

$$\int x\,e^x\,dx = \int u\,dv = u\,v - \int v\,du = x\,e^x - \int e^x\,dx\ = x\,e^x - e^x + C$$

The same thing, of course! Just a different approach, which some folks like better.

Questions

Consider the integral $\int x\cos(2\,x)\,dx$.

I see a **product**, x and cos(2x). I don't mind differentiating or anti-differentiating either one of these function. However, if I differentiate x it becomes 1; that will make for a nicer product. That suggests I choose f(x)=x, and g'(x)=cos(2x).

- What is $\int x\cos(2\,x)\,dx$ if you use integration by parts with $u = x$ and $dv = \cos(2\,x)\,dx$?
- What would happen if you try to use integration by parts with $u = \cos(2\,x)$ and $dv = x\,dx$?

(Video)

Question

How can you use integration by parts to evaluate $\int x^2\sin(3\,x)\,dx$? (Video)

Question

How can you use integration by parts to evaluate $\int \tan^{-1}(x)\,dx$? (Video)

Guidelines

Guidelines for choosing *u* and *dv*

- You have to be able to integrate dv (that is, know the anti-derivative v).
- The derivative of u and the integral of dv can't get too much messier than they already are. You want $\int v\,du$ to be no worse than $\int u\,dv$.

Questions

- To evaluate $\int x^n\ln(x)\,dx$, use integration by parts with $u = \ln(x)$ and $dv = x^n\,dx$. (Video)

- Using your results from the previous question, what is $\int \ln(x)\,dx$? (Video)
- Evaluate $\int_0^\pi x^2 e^{-4x}\,dx$ using integration by parts. (Video)
- Evaluate $\int x^3 \cos(x^2)\,dx$ using integration by parts, but first make a substitution. (Video)
- What is the area of the region bounded by $y = \sin^{-1}(x/2)$, the x-axis, and $x = 1$? (Video)
- What is the volume of the solid obtained by rotating about the x-axis the region bounded by $y = x\ln(x)$ and the x-axis for $1 \le x \le e$. (Video)
- In the first question above, there is one special case: $n = -1$. Use integration by parts in this particular instance to get $\int x^{-1} \ln(x)\,dx = \text{stuff} - \int x^{-1} \ln(x)\,dx$. Solve this equation for $\int x^{-1} \ln(x)\,dx$ to finish evaluating the integral (**what a great trick!**). (Video)
- To evaluate $\int e^x \cos(x)\,dx$, use integration by parts **twice**. (Be sure to choose u and dv the same way both times. If you choose $u = e^x$ the first time, be sure to choose $u = e^x$ the second time. Or, if you choose u = trig the first time, choose u = trig the second time.) Then employ what you did in the last problem to finish evaluating the integral. (Video)

Homework

IMath problems on integration by parts.