Basic Integration

MAT 229, Spring 2021

Week 4

There are two halves to calculus, differentiation and integration. Integration gives us the ability to determine quantities like area and volume. Plus, it plays a major role in determining behavior when all we know is how that behavior changes.

Calculus I introduced you to integration and we have added a few integral formulas in the first three weeks of our class. For the next several weeks we will explore integration more deeply from a variety of perspectives:

- Techniques of integration
- Numerical integration.
- Improper integrals

This section is a summary of integration to date.

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  - Section 4.4: Indefinite integrals
  - Section 4.5: The substitution rule
- Boelkins/Austin/Schlicker's Active Calculus
  - Section 5.2: The second fundamental theorem of calculus
  - Section 5.3: Integration by substitution

Basic integrals

We have several basic differentiation formulas. Reversing these gives us the basic integral formulas. Check the validity of each formula by differentiating the right side. You should get the integrand of the integral on the left.

- Power rule: $\int x^n \, dx = \left\{ \begin{array}{ll} \frac{1}{n+1} x^{n+1} & \text{if} \quad n \neq -1 \\ \ln |x| & \text{if} \quad n = -1 \end{array} \right\} + C$
- Exponential rule: $\int e^{rx} \, dx = \frac{1}{r} e^{rx} + C$

- Trigonometric rules:
  - $\int \cos(a\,x)\,dx = \frac{1}{a}\sin(a\,x) + C$
  - $\int \sin(a\,x)\,dx = -\frac{1}{a}\cos(a\,x) + C$
  - $\int \sec^2(a\,x)\,dx = \frac{1}{a}\tan(a\,x) + C$
  - $\int \csc^2(a\,x)\,dx = -\frac{1}{a}\cot(a\,x) + C$
  - $\int \sec(a\,x)\tan(a\,x)\,dx = \frac{1}{a}\sec(a\,x) + C$
  - $\int \csc(a\,x)\cot(a\,x)\,dx = -\frac{1}{a}\csc(a\,x) + C$
- Rules from inverse trigonometry derivatives:
  - $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(x/a) + C$
  - $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(x/a) + C$

Example

$$\int\left(5\,e^{2\,x} - \frac{11}{\sqrt{9-x^2}} + 8\sec^2(7\,x)\right)dx = 5\int e^{2\,x}\,dx - 11\int\frac{1}{\sqrt{3^2-x^2}}\,dx + 8\int \sec^2(7\,x)\,dx$$
$$= \frac{5}{2}e^{2\,x} - 33\sin^{-1}(x/3) + \frac{8}{7}\tan(7\,x) + C$$

Questions

Use basic integral formulas to evaluate the following integrals.

- $\int\left(\frac{7}{x} - \frac{5}{7+x^2}\right)dx$ (Video)
- $\int_0^1\left(2\sqrt{x} + \sec\left(\frac{\pi}{4}x\right)\tan\left(\frac{\pi}{4}x\right)\right)dx$ (Video)

Substitution

Every integration rule is a differentiation rule backwards. There is a beautiful symmetry between integration and differentiation. Substitution is the chain rule in reverse.

You'll recall that the chain rule is all about differentiating compositions; so the substitution rule is all about identifying the interior function in a composition (which we frequently call *u(x)*).

The chain rule

$$(F(u(x)))' = F'(u(x))\,u'(x)$$

in reverse with $f(x) = F'(x)$ produces the substitution rule.

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$

The key to recognizing a good substitution $u(x)$ for an integral is locating $u'(x)$ in the integrand.

Example

To find $\int \tan(x)\,dx$, write it as $\int \frac{\sin(x)}{\cos(x)}\,dx$. Note that $\sin(x)$ is almost the derivative of $\cos(x)$. Try the substitution

$$u = \cos(x) \longrightarrow du = -\sin(x)\,dx \ \text{ or } \ -du = \sin(x)\,dx$$

Evaluate the integral with this substitution.

$$\int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{-1}{u}\,du = -\ln|u| + C = -\ln|\cos(x)| + C$$

An alternate form for this integral comes from properties of logarithms.

$$-\ln|\cos(x)| + C = \ln|\cos(x)^{-1}| + C = \ln|\sec(x)| + C$$

Questions

Use appropriate substitutions to evaluate the following integrals.

- $\int \cot(x)\,dx$ (Video)
- $\int \sin(2x)\,e^{\cos(2x)}\,dx$ (Video)
- $\int \frac{x}{4+x^4}\,dx$ (Video)