Taylor Polynomials

MAT 229, Spring 2021

Week 8

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus*
  11.11: Taylor Polynomials
- Boelkins/Austin/Schlicker's Active Calculus
  8.5 Taylor Polynomials and Taylor Series

Tangent line approximation

Questions

Let $f(x) = \tan^{-1}(3x)$.

- What is an equation of the tangent line to $y = f(x)$ at $x = 0$?
- What is the tangent line approximation to $f(x)$ at 0?

(Video)

Formula

Given a general function $g(x)$ and value $x = a$, the tangent line approximation to $g(x)$ at a is

$$T(x) = g(a) + g'(a)(x - a)$$

Questions

- How does $T(a)$ compare to $g(a)$?
- How does $T'(a)$ compare to $g'(a)$?

(Video)

Quadratic approximation

The quadratic approximation to function $g(x)$ at $x = a$ is the quadratic polynomial function $T_2(x) = P x^2 + Q x + R$ such that

- $T_2(a) = g(a)$
- $T_2'(a) = g'(a)$
- $T_2''(a) = g''(a)$

It is a little easier to work with $T_2(x)$ when it has the form

$$T_2(x) = A + B(x - a) + C(x - a)^2$$

Questions

Consider $f(x) = \cos(x) + \sin(2x)$. We want to find its quadratic approximation when $a = 0$. Let $T_2(x) = A + B(x - a) + C(x - a)^2$

- What does $T_2(0) = f(0)$ tell us about A, B, C?
- What does $T_2'(0) = f'(0)$ tell us about A, B, C?
- What does $T_2''(0) = f''(0)$ tell us about A, B, C?
- Plot $y = f(x)$ and $y = A + B x + C x^2$ together on the same coordinate axes and compare them.

(Video)

Questions

Consider $g(x) = x \ln(x)$. We want to find its quadratic approximation when $a = 1$. Let $T_2(x) = A + B(x - a) + C(x - a)^2 = A + B(x - 1) + C(x - 1)^2$

- Find the values of A, B, C that make $T_2(1) = g(1)$, $T_2'(1) = g'(1)$, $T_2''(1) = g''(1)$
- What is the absolute error in approximating $g(0.8)$ with $T_2(0.8)$?
- What is the absolute error in approximating $g(1.5)$ with $T_2(1.5)$?

(Video)

Taylor polynomial approximation

Notation

For the next month we will be taking higher order derivatives. Denote the k^{th} derivative of $f(x)$ as $f^{(k)}(x)$.

Factorials

If k is a whole number we say the **factorial** of k, denoted $k!$, is the quantity

$$k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (k - 1) \cdot k$$

Because it is useful to do so, define $0! = 1$.

Examples

- $0\,! = 1$
- $1! = 1$
- $2! = 1 \cdot 2 = 2$
- $3! = 1 \cdot 2 \cdot 3 = 6$
- $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$
- $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

Taylor polynomials

The n^{th} degree Taylor polynomial approximation of $f(x)$ at $x = a$ is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f^{(3)}(a)(x-a)^3 + \ldots + \frac{1}{(n-1)!} f^{(n-1)}(a)(x-a)^{n-1} + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Notes

- The first degree Taylor polynomial approximation is the linear approximation.
- The second degree Taylor polynomial approximation is the quadratic approximation.

Questions

- What is the fourth degree Taylor polynomial for $f(x) = e^x$ at $a = 0$?
- Approximate $e^{0.5}$ with this Taylor polynomial. What is the absolute error in this approximation?
- Plot it along with $y = e^x$.

(Video)

Questions

- What is the fifth degree Taylor polynomial for $g(x) = \cos(x)$ at $a = \pi$? Plot it along with $y = \cos(x)$.
- From this graph, estimate for which range of values of x, this Taylor polynomial approximates $\cos(x)$ with error less than 0.1.

(Video)

Questions

Given a general differentiable function $f(x)$. Reminder:

The n^{th} degree Taylor polynomial approximation of $f(x)$ at $x = a$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f^{(3)}(a)(x-a)^3 + \ldots + \frac{1}{(n-1)!} f^{(n-1)}(a)(x-a)^{n-1} + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

- How does $T_n(a)$ compare with $f(a)$?
- How does $T_n{}^{(1)}(a)$ compare with $f^{(1)}(a)$?
- How does $T_n{}^{(2)}(a)$ compare with $f^{(2)}(a)$?
- How does $T_n{}^{(3)}(a)$ compare with $f^{(3)}(a)$?

(Video)

Homework

- IMath problems on Taylor Polynomials.