Weekly Assignment 10

MAT 229, Spring 2021

Instructions: **Show your work!**

1. Let $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^n}{2^n\, n}$

- What is the domain of $f(x)$? In other words, what is the interval of convergence for the power series?

In[696]:=
```
Clear[a, b, n]
(* Let's integrate a term, to see what the new terms look like: *)
a[n_] = (-1) ^ (n + 1) (x - 2) ^ n / (2 ^ n n)
ratio = Simplify[Abs[a[n + 1] / a[n]]]
Limit[ratio, n → Infinity]
```

Out[697]= $\frac{(-1)^{1+n}\, 2^{-n}\, (-2+x)^n}{n}$

Out[698]= $\frac{1}{2} \mathrm{Abs}\left[\frac{n\,(-2+x)}{1+n}\right]$

Out[699]= $\frac{1}{2} \mathrm{Abs}[-2+x]$

ratio test: $\frac{|x-2|}{2} < 1 \rightarrow |x-2| <$

So as long as |x-2|<2, this converges. Obviously 0 doesn't work, since the series is effectively harmonic; and at 4 we get the alternating harmonic, which converges. Hence the domain is (0,4].

In[700]:=
```
f[x_] := Sum[a[n], {n, 1, Infinity}]
f[x]
```

Out[701]= $-\mathrm{Log}[2] + \mathrm{Log}[x]$

$0 \rightarrow (-1)^{n+1} \frac{(-2)^n}{2^n\, n}$

$-(-1)\frac{1}{n}$ X

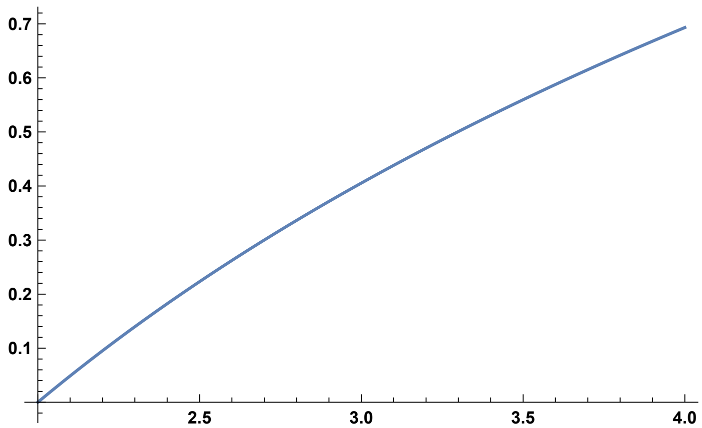
$4 \rightarrow (-1)^{n+1} \frac{(2^n)}{2^n\, n}$

$= (-1)^{n+1} \frac{1}{n}$ ✓

- Approximate $\int_2^4 f(x)\, dx$ with error less than 0.0001.

In[702]:=
```
Plot[f[x], {x, 2, 4}]
```

Out[702]=



In[703]:=
```
(* Let's integrate a term, to see what the new terms look like: *)
b[n_] = Integrate[a[n], x]
Sum[b[n], {n, 1, Infinity}]
```

Out[703]= $\frac{(-1)^{1+n}\, 2^{-n}\, (-2+x)^{1+n}}{n\,(1+n)}$

Out[704]= $2 - x + x \, \text{Log}\left[\frac{x}{2}\right]$

This is an antiderivative.

These are the terms obtained by antidifferentiating term by term. They alternate, we need to evaluate at 4 then at 2

$\frac{i^n 2^{n+1}}{n(n+1)}$

0

In[705]:=
```
Integrate[Sum[a[n], {n, 1, Infinity}], x]
solns = NSolve[2/((2+n) (1+n)) == .0001, n]
nterms = Ceiling[n /. solns[[2]]];
Sum[2 (-1)^(1+n) / (n (1+n)), {n, 1, nterms}];
approx = N[%];
true = NIntegrate[f[x], {x, 2, 4}];
absError = Abs[approx - true];
TableForm[
 Transpose[{{true, approx, nterms, absError}}]
 , TableHeadings -> {{"True", "Partial", "No. of terms", "Abs. Error"}}
]
```

$(n+1)^{th}$ term $< .0001$ $(n=140)$

Out[705]= $-x + x \, \text{Log}\left[\frac{x}{2}\right]$

This is another antiderivative.

Just wanted to contrast what we got antidifferentiating the function versus the Taylor series.

Out[706]= {{n → -142.922}, {n → 139.922}}

Out[712]//TableForm=

True	0.772589
Partial	0.772538
No. of terms	140
Abs. Error	0.000050298

✓

2. Let $g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi)^{2n}}{(2n)!}$

- What is the domain of $g(x)$? In other words, what is the interval of convergence for the power series?

$\mathbb{R}$ - all real numbers

by the ratio test:

In[713]:=
```
Clear[a, b, n]
(* Let's integrate a term, to see what the new terms look like: *)
a[n_] = (-1) ^ (n) (x - π) ^ (2 n) / Factorial[2 n]
ratio = Simplify[Abs[a[n + 1] / a[n]]]
Limit[ratio, n → Infinity]
```

Out[714]= $\frac{(-1)^n (-\pi + x)^{2n}}{(2n)!}$

Out[715]= $\mathrm{Abs}\left[\frac{(\pi - x)^2 (2n)!}{(2(1+n))!}\right]$

Out[716]= 0

So this converges everywhere. The factorials make it so....

In[717]:=
```
g[x_] := Sum[a[n], {n, 0, Infinity}]
g[x]
```

Out[718]= $-\mathrm{Cos}[x]$ ← Hey, it's cosine!

- Find a power series representation for $g'(x)$.

Differentiate term-by-term

In[719]:=
```
Clear[b, n]
b[n_] = (-1) ^ (n) D[(x - π) ^ (2 n), x] / Factorial[2 n]
Sum[b[n], {n, 1, Infinity}]
Simplify[%]
```

Sum those up, &
what do you get?

Out[720]= $\frac{2 (-1)^n n (-\pi + x)^{-1+2n}}{(2n)!}$

Out[721]= $-\frac{(\pi - x)\,\mathrm{Sin}[x]}{-\pi + x}$

Out[722]= $\boxed{\mathrm{Sin}[x]}$

Just as we'd expect — the derivative of cosine.

3. Let $h(x) = x \ln(1 + x)$

- Using geometric series, what is a power series representation for $h(x)$ centered at 0?

$$\frac{d}{dx} \ln(1+x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

In[723]:=
```
Clear[n]
h[x_] := x Log[1 + x]
Series[h[x], {x, 0, 10}]
```

Out[725]= $x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \frac{x^7}{6} + \frac{x^8}{7} - \frac{x^9}{8} + \frac{x^{10}}{9} + O[x]^{11}$

- Using the power series representation, approximate $\int_0^1 h(x)\,dx$ with error less than 0.0001.

In[726]:=
```
(* Let's integrate a term: *)
(-1) ^n Integrate[x^n, x] / (n - 1)
Sum[(-1) ^n Integrate[x^n, x] / (n - 1), {n, 2, Infinity}]
```

$$\ln(1+x) = \sum_{n=0} \int (-x)^n\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C \qquad C=0$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad ✓$$

$$\therefore\ h(x) = x \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n}$$

Out[726]= $\frac{(-1)^n x^{1+n}}{(-1+n)\,(1+n)}$

Out[727]= $\frac{1}{4}\left(2\,x - x^2 - 2\,\mathrm{Log}[1+x] + 2\,x^2\,\mathrm{Log}[1+x]\right)$

$$\int_0^1 h(x)\,dx = \sum_{n=1}^{\infty} (-1)^{n+1} \int_0^1 \frac{x^{n+1}}{n}\,dx$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \left.\frac{x^{n+2}}{n(n+2)}\right|_0^1$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+2)}$$

In[728]:=

```
soln = NSolve[1/((n+1)^2-1) == .0001, n]
nterms = Ceiling[n /. soln[[2]]]
partial = Sum[(-1)^n/((-1+n) (1+n)), {n, 2, nterms}];
true = NIntegrate[h[x], {x, 0, 1}];
absError = Abs[true - partial];

TableForm[
 Transpose[{{true, N[partial], nterms, absError}}]
 , TableHeadings -> {{"True", "Partial", "No. of terms", "Abs. Error"}}
]
```

Alternating series,

Out[728]= {{n → -101.005}, {n → 99.005}}

Out[729]= 100

Out[733]//TableForm=

True	0.25
Partial	0.25005
No. of terms	100
Abs. Error	0.000049505

Find n of 1st unselected term < .0001 :

$$\frac{1}{(n+1)(n+3)} < .0001 \rightarrow$$

note s_n; $n \approx 98$; 1st term for n

Note: I indexed from 1 in the handwritten stuff, but from 2 in the Mathematica: 99 $\boxed{\text{terms}}$ either way...