Weekly Assignment #6

Instructions: work should be done by hand, when possible, but use technology to confirm your answers. **Show your work!**

1. Improper integral: infinite interval of integration

Let $f(x) = x(e^{-2x} + e^{-3x})$.

a. Consider the integral $A = \int_0^\infty f(x)\, dx$. Rewrite the integral as a limit of a proper integral.

$\boldsymbol{A = \int_0^\infty f(x)\, dx = \lim\{R\text{->}\infty\}\left(\int_0^R f(x)\, dx\right)}$

b. Evaluate the integral A, as a limit.

In[107]:=
```
Integrate[x (E^(-2 x) + E^(-3 x)), {x, 0, Infinity}]
Limit[Integrate[x (E^(-2 x) + E^(-3 x)), {x, 0, r}], r → Infinity]
```

Out[107]= $\frac{13}{36}$

Out[108]= $\frac{13}{36}$

2. A surprising approach to integration by parts

In the most recent lab, I was surprised to see some folks' approach to a particular integral. You were to compute the integral $\int_1^e x\,(\ln(x))^2\, dx$. Some of you chose to begin by a substitution, to rewrite the integral prior to integration by parts. Several students began with "exponential substitution", $x = e^u$, hence $\ln(x) = u$, and dx = e^u du. This gave rise to the integral $B = \int_0^1 e^{2u}\, u^2\, du$ (note the change to the limits). Then they did an integration by parts from there.

Suppose that we had wanted to compute $B = \int_0^e x\,(\ln(x))^2\, dx$ instead. This integral is improper.

a. If you make the same substitution as above, the only thing that changes is the limits. Write this new integral, which has the same value as B.

$\boldsymbol{B = \text{Limit}\,\{r \to \infty\}\,\left(\int_{-r}^1 e^{2u}\, u^2\, du\right)}$ $= \int_{-\infty}^{1} e^{2u} u^2 du = I$

b. Find the value of these improper integrals, by treating either one (your choice! They have the same value....) as a limit.

$f(u) = u^2 \quad g'(u) = e^{2u}$

$f'(u) = 2u \quad g(u) = \frac{1}{2} e^{2u}$

$I = \frac{u^2}{2} e^{2u}\Big|_{-\infty}^{1} - \int_{-\infty}^{1} u e^{2u} du = \frac{e^2}{2} - \int \frac{u}{2} e^{2u}\Big|_{-\infty}^{1} -$

$\int_{-\infty}^{1} \left(\frac{1}{2} e^{2u} \ldots \right) = \frac{e^2}{2} - \frac{e^2}{2} +$

$\frac{1}{4} e^{2u} \Big|_{-\infty}^{1} = \frac{e^2}{4}$

≈ 1.947

```
true = Integrate[x Log[x]^2, {x, 0, E}]
N[%]
Limit[Integrate[x Log[x]^2, {x, r, E}], r → 0]
Integrate[u^2 E^(2 u), {u, -Infinity, 1}]
N[%]
Limit[Integrate[u^2 E^(2 u), {u, -r, 1}], r → Infinity]
```

Out[126]= $\frac{e^2}{4}$

Out[127]= 1.84726

Out[128]= Indeterminate

Out[129]= $\frac{e^2}{4}$

Out[130]= 1.84726

Out[131]= $\frac{e^2}{4}$

c. Estimate the value of B (in its given form) using the midpoint rule with n=1000 (M_{1000}), and compare to the actual value computed in part 2.

```
f[x_] := x (Log[x])^2
a = 0.0
b = E
n = 1000
dx = (b - a) / n
mid = dx * Sum[f[a + (k - 0.5) dx], {k, 1, n}]
{N[true], mid, mid - true}
```

Out[140]= 0.

Out[141]= e

Out[142]= 1000

Out[143]= 0.00271828

Out[144]= 1.84728

Out[145]= {1.84726, 1.84728, 0.0000144506}

Midpoint is an overestimate, but is pretty close!

d. Explain why it is impossible to use the error estimate for the midpoint rule in this case. Why is it impossible to use Simpson's rule?

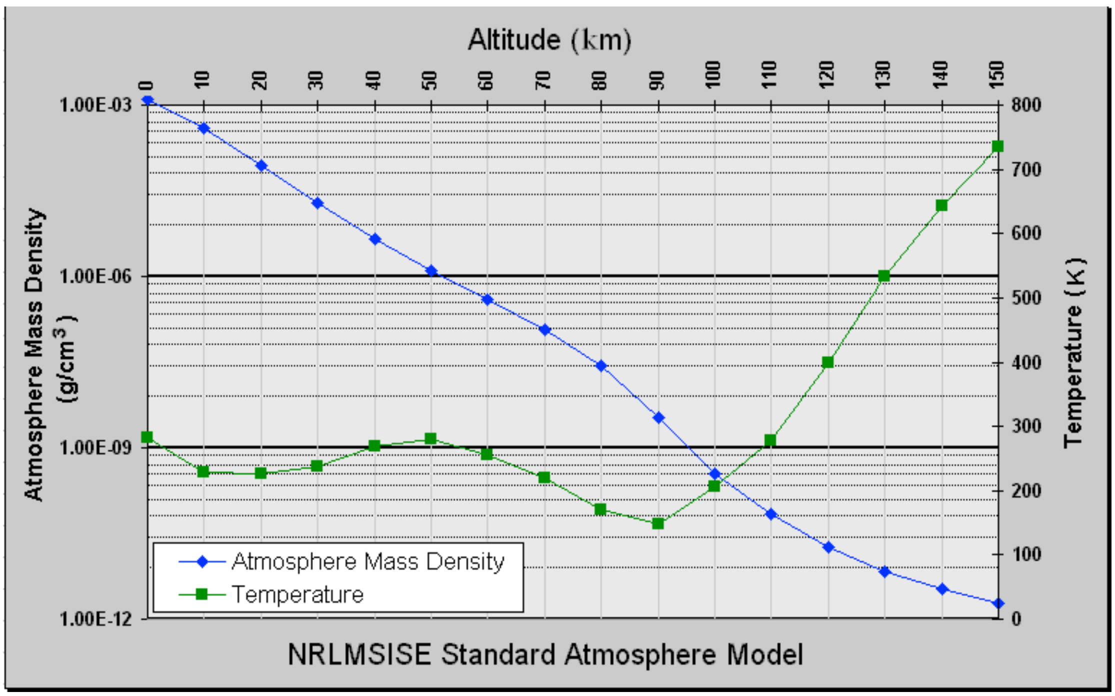
We can't use the endpoint at 0 in a calculation, because the integrand is infinite there (that shoots Simpson's out of the water). We can't create an error estimate for midpoint, because the second derivative is unbounded.

3. Application

There is a shell of air around the Earth (whose radius is 6,360 km), and the mass density of this shell decreases with height, tending toward zero as the height goes to ∞.

Let's compute the mass of the Earth's atmosphere. (According to the National Center for Atmospheric Research (NCAR), "The total mean mass of the atmosphere is 5.1480 $\times 10^{18}$ kg....") That's pretty heavy, and it's weighing on you and me all the time! He ain't heavy; he's my atmosphere.

This plot shows us how the Earth's mass density varies as a function of altitude, on a "log scale" (in the following discussion we ignore the Temperature -- focus on the blue!):

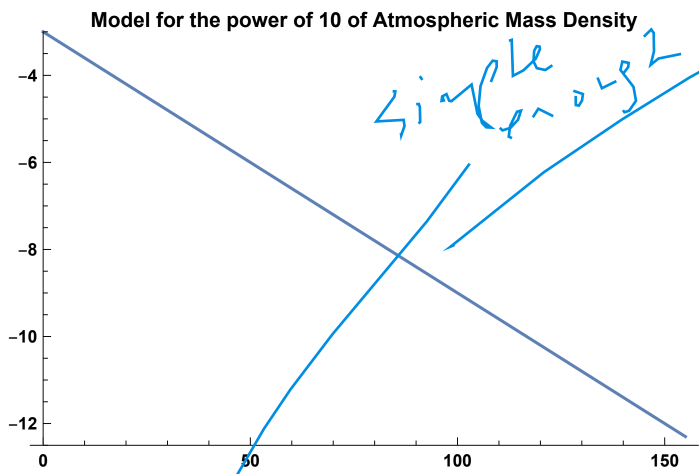


1. We notice that, on the log scale, the density is roughly linear. Draw a straight line through the blue points that fits the data pretty well.

2. Notice that the y-axis can be thought of as "powers of 10" (and they're getting more negative). Think of y as -3, -6, -9, and -12. So if our line is *y=mx+b*, let's say *y=-3(1+x/50)* -- which I obtained by passing a line through the points (0,-3) and (150,-12) -- then the model for density ρ (in units of g/cm^3) becomes

```
ρ[x_] := 10^(-3 (1 + x/50))
```

Simple enough to change from ba



use base e for calculus

$-\ln(10)\,3(1+x/50)$

Rewrite this function $\rho(x)$ using base E, instead of base 10.

```
ρ[x_] := (E^Log[10])^(-3 (1 + x / 50))
```

$= e$

3. To compute the mass of the atmosphere, we have to multiply the density (which has units mass per volume) times a lot of tiny volumes (dV). Each little volume is a spherical shell at a height x above the surface of the Earth. Since the Earth has radius 6,360 km, the shells looks like dV(x)=4 $\pi(x + 6360)^2$dx

Compute the improper integral

$$\int_0^\infty \rho(x)\, dV(x)$$

as a limit.

```
In[154]:= mass = Integrate[ρ[x] 4 π (x + 6360)^2, {x, 0, Infinity}]
          N[%]
          Limit[Integrate[ρ[x] 4 π (x + 6360)^2, {x, 0, r}], r → Infinity]
Out[154]= (40 π (25 + 1908 Log[10] (5 + 954 Log[10]))) / (27 Log[10]^3)
Out[155]= 3.68762 × 10^6
Out[156]= (40 π (25 + 1908 Log[10] (5 + 954 Log[10]))) / (27 Log[10]^3)
```

4. That answer is in the units of "$km^3 g/cm^3$"; we want it in kilograms, to compare to NCAR's answer. Do the unit conversion (km to meters, cm to meters, g to kg). How close are we to their answer of 5.1480 $\times 10^{18}$ kg?

```
In[ ]:= mass = N[(1000)^2 * (100)^3 * mass]
Out[ ]= 3.68762 × 10^18
```

Not nearly as close as I'd like to be; but "within an order of magnitude", as the physicists like to say. I guess we'd need to make a little better model for the mass density than a straight line!