Weekly Assignment 11

MAT 229, Spring 2021

Instructions: **Show your work!**

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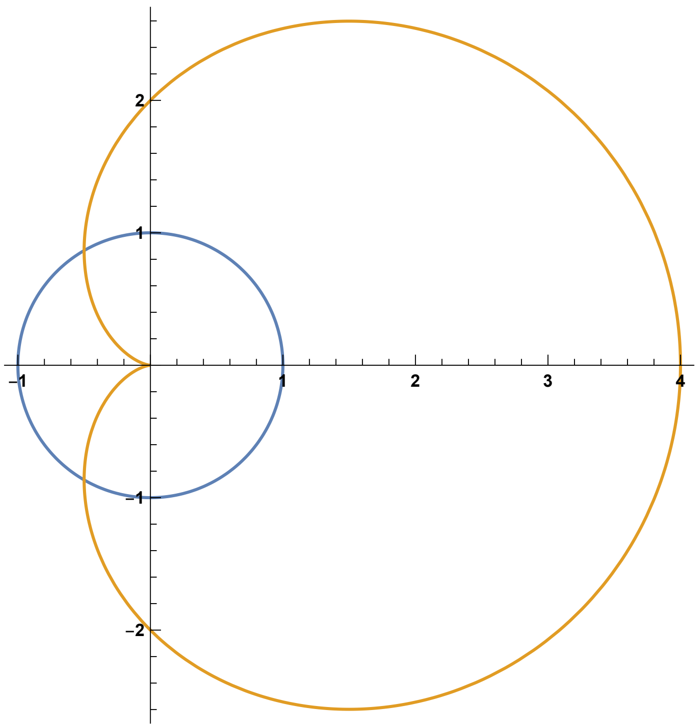
1. Polar curves

A. Consider the two polar curves $r = 1$ and $r = 2 + 2\cos(\theta)$.

```
PolarPlot[{1, 2 + 2 Cos[theta]}, {theta, 0, 2 Pi}]
```

Out[ ]=


a. Find polar coordinates for all points of intersection.

```
Solve[2 + 2 Cos[θ] == 1, θ,]
(* 2π/3, -2π/3 *)
```

Out[ ]= $\{\{\theta \to \text{ConditionalExpression}[-\frac{2\pi}{3} + 2\pi C[1], C[1] \in \mathbb{Z}]\}, \{\theta \to \text{ConditionalExpression}[\frac{2\pi}{3} + 2\pi C[1], C[1] \in \mathbb{Z}]\}\}$

b. Find the area of the region that is inside $r = 2 + 2\cos(\theta)$ and outside $r = 1$.

In[∘]:=
```
Integrate[1/2 ((2 + 2 Cos[th])^2 - 1), {th, -2π/3, 2π/3}]
N[%]
```

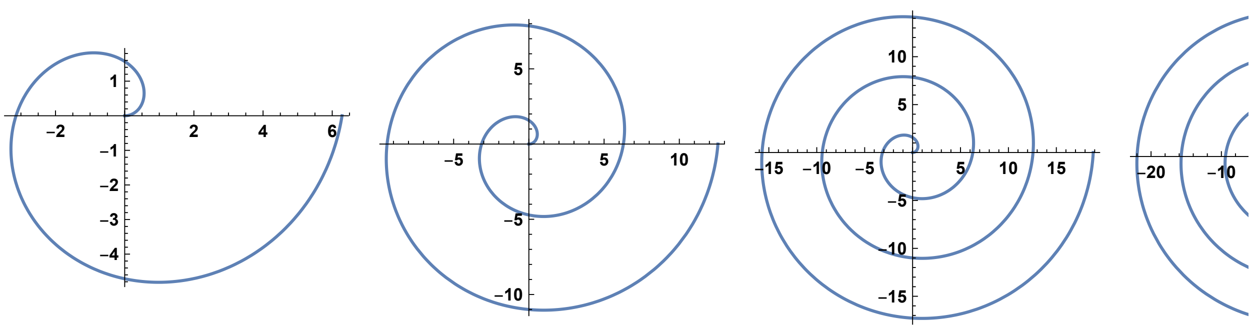
Out[∘]= $\frac{7\sqrt{3}}{2} + \frac{10\pi}{3}$

Out[∘]= 16.5342

B. Consider the polar curve $r = \theta$ (An Archimedean spiral).

```
(* If you ask me to consider a certain polar curve,
 the first thing I want to do is plot it!:) *)
GraphicsGrid[{{
   PolarPlot[{theta}, {theta, 0, 2 Pi}],
   PolarPlot[{theta}, {theta, 0, 2 * 2 Pi}],
   PolarPlot[{theta}, {theta, 0, 3 * 2 Pi}],
   PolarPlot[{theta}, {theta, 0, 4 * 2 Pi}],
   PolarPlot[{theta}, {theta, 0, 5 * 2 Pi}]
  }}]
```

Out[∘]=



a. Find the length of the curve swept out after n complete rotations from angle 0; your answer should be a formula involving n.

```
(*
This is what our formula gives
 us: notice that n complete rotations is n*2Pi radians....
    I've also added some assumptions:
  this is nice when Mathematica is trying to handle a lot of "irrelevant"
   cases for me. My n is a natural number (that is, a positive integer):
*)
length[n_] =
 Integrate[Sqrt[th^2 + 1], {th, 0, 2 Pi n}, Assumptions → n ∈ Integers && n > 0]
(* That's a very cool looking formula! *)
```

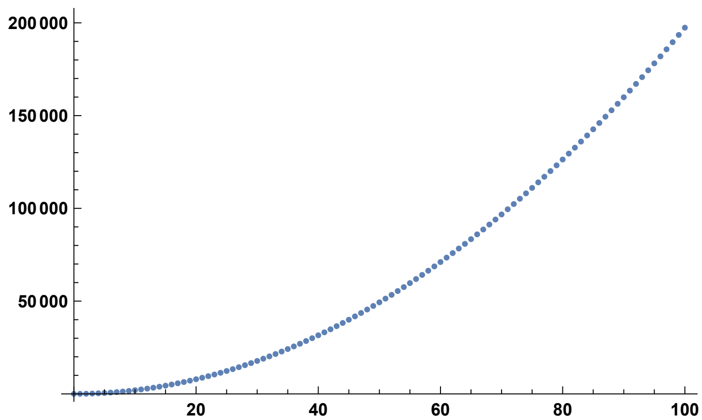
Out[ ]= $n \pi \sqrt{1 + 4 n^2 \pi^2} + \frac{1}{2} \text{ArcSinh}[2 n \pi]$

```
(* I'm going to compare my calculated lengths with what I'd
 get by doing a sum of a bunch of circles of the average radius,
because they should be rather close: *)
tab = Table[{
    n,
    N[length[n]], (* Here are the actual lengths *)
    Sum[2.0 Pi (2 m - 1) Pi, {m, 1, n}]
    (* This is the sum of a bunch of circumferences with average radii....*)
   }, {n, 1, 10}
  ];
TableForm[tab, TableHeadings → {{}, {"n", "exact", "approximated"}}
]
(* pretty close!  It's interesting that the differences are so consistent.... *)
```

Out[ ]//TableForm=

n	exact	approximated
1	21.2563	19.7392
2	80.8193	78.9568
3	179.718	177.653
4	318.036	315.827
5	495.801	493.48
6	713.023	710.612
7	969.71	967.221
8	1265.86	1263.31
9	1601.49	1598.88
10	1976.59	1973.92

Out[ ]=



b. Find the **area** of the region that is swept out over the **last** complete rotation when using *n* complete rotations from angle 0; your answer should be a formula involving *n*. (You might use a circle as an approximation to check your answer.)

```
Integrate[1/2 (th^2), {th, 0, 1*2 Pi }, Assumptions → n ∈ Integers && n > 0]
Integrate[1/2 (th^2), {th, 1*2 Pi, 2*2 Pi }, Assumptions → n ∈ Integers && n > 0]
area[n_] = Integrate[1/2 (th^2),
   {th, (n - 1) * 2 Pi, n * 2 Pi }, Assumptions → n ∈ Integers && n > 0]
(* I'm going to compare my calculated areas with the area of a circle
 of the average radius, because they should be rather close: *)
tab = Table[{
     n,
     N[area[n]], (* Here are the actual lengths *)
     N[Pi ((2 n - 1) Pi) ^2]
      (* This is the area of a circle with average radii....*)
    }, {n, 1, 10}
   ];
TableForm[tab, TableHeadings → {{}, {"n", "exact", "approximated"}}
]
(* pretty close! It's interesting that the
   differences are so consistently equal to about 10.... *)
```

Out[•]= $\frac{4\pi^3}{3}$

Out[•]= $\frac{28\pi^3}{3}$

Out[•]= $\frac{1}{2}\left(-\frac{8}{3}(-1+n)^3\pi^3+\frac{8n^3\pi^3}{3}\right)$

Out[•]//TableForm=

n	exact	approximated
1	41.3417	31.0063
2	289.392	279.056
3	785.492	775.157
4	1529.64	1519.31
5	2521.84	2511.51
6	3762.09	3751.76
7	5250.4	5240.06
8	6986.75	6976.41
9	8971.15	8960.81
10	11 203.6	11 193.3

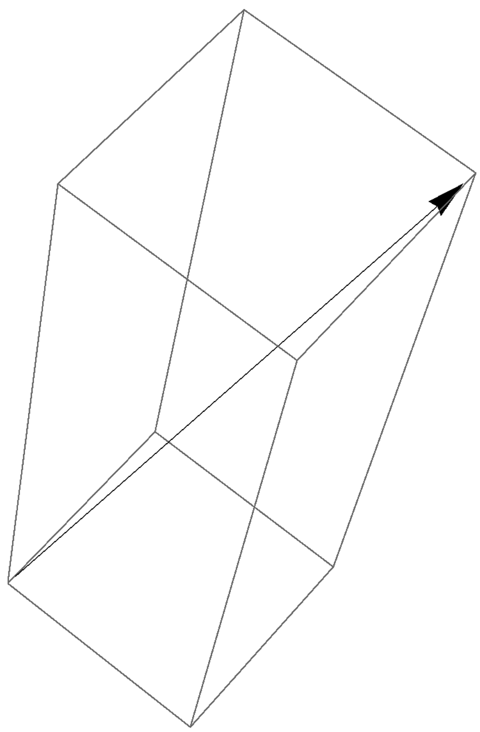
2. Shapes in space

Let P be the point with Cartesian coordinates (2, 1, 4) and Q be the point (4, 3, 10).

In[3]:=
```
p = {2, 1, 4}
q = {4, 3, 10}
Graphics3D[Arrow[{p, q}]]
```

Out[3]= {2, 1, 4}

Out[4]= {4, 3, 10}

Out[5]=


a. What is the distance between them?

In[•]:=
```
Norm[q - p]
N[%]
```

Out[•]= $2\sqrt{11}$

Out[•]= 6.63325

b. What are the coordinates for the midpoint of the line segment $\overline{PQ}$?

In[6]:=
```
midpoint = p + 1/2 (q - p)
radius = 1/2 Norm[q - p]
```

Out[6]= {3, 2, 7}

Out[7]= $\sqrt{11}$

c. Find an equation for the sphere that has a diameter with one endpoint at *P* and the other at *Q*.

In[8]:=
```
Norm[{x, y, z} - midpoint]^2 == radius^2
```

Out[8]= $\text{Abs}[-3+x]^2 + \text{Abs}[-2+y]^2 + \text{Abs}[-7+z]^2 == 11$

3. Unit vectors

a. Find the two unit vectors that are parallel to vector ⟨2, 6⟩.

In[•]:=
```
u = {2, 6}
uhat = u / Norm[u]
-uhat
```

Out[•]= {2, 6}

Out[•]= $\left\{\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right\}$

Out[•]= $\left\{-\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right\}$