Section Summary: The dot product

a. Definitions

If **a**= $\langle a_1, a_2, a_3\rangle$ and **b**= $\langle b_1, b_2, b_3\rangle$, then the **dot product** of **a** and **b** is the number $\mathbf{a}\cdot\mathbf{b}$ given by

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Two non-zero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \frac{\pi}{2}$.

The **direction angles** of a non-zero vector **a** are the angles α, β, and γ that the vector makes with the positive x-, y-, and z-axes. The cosines of these angles are called the **direction cosines**.

The **vector projection** of **b** onto **a** is

$$\text{proj}_a\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|}$$

The **scalar projection** of **b** onto **a** is

$$\text{comp}_a\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

In particular, the **unit vector** created from **a** would be

$$\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

(the vector with the same direction, but unit length).

b. Theorems

If θ is the angle between vectors **a** and **b**, then

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Corollary:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Hence, **a** and **b** are orthogonal (i.e. perpendicular) $\iff$ $\mathbf{a}\cdot\mathbf{b} = 0$

c. Properties/Tricks/Hints/Etc.

Properties of the dot product:

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

d. Summary

The dot product is a way of combining two vectors to get a scalar (i.e., a number). It effectively measures the shadow that one vector casts on the other, and if the dot product is zero, the two vectors are orthogonal (i.e. perpendicular): neither vector casts a shadow on the other.