Section Summary 6.3 - logarithmic functions

a. Definitions

- **logarithmic function with base b**:

$$\log_b x = y \iff b^y = x$$

The logarithm is defined as the inverse function of the exponential function, each with base b. It is the primary motivation for starting off this course with inverses: other inverses may be interesting, but this one is essential!

- **natural log**: the logarithmic function with base e, denoted ln.

b. Theorems

- The cancellation laws in this particular case lead to the form

$$\log_b(b^x) = x$$

for all $x \in \Re$, and

$$b^{\log_b x} = x$$

for all $x > 0$. The most important example of these is the natural log and exponential functions, where

$$\ln(e^x) = x$$

for all $x \in \Re$, and

$$e^{\ln x} = x$$

for all $x > 0$.

- The properties of logarithms are **reflections** (across the line $y = x$) of the properties of exponents:

i. "The log of a product is the sum of the logs":

$$\ln(xy) = \ln x + \ln y$$

ii.

$$\ln(x^r) = r \ln x$$

iii. "The log of a quotient is the difference of the logs"

$$\ln(\frac{x}{y}) = \ln x - \ln y$$

(this one is a combination of the two preceding rules).

c. Properties/Tricks/Hints/Etc.

- It is possible to turn any exponential with base b into a natural logarithmic function: we can find β such that $e^{\beta} = b$ by using the inverse property of the logarithm:

  $$\ln b = \beta$$

  Hence,

  $$b^x = (e^{\ln(b)})^x$$

  and using a property of exponents,

  $$b^x = e^{\ln(b)x}$$

  Now we can forget about other bases (hooray!), and work only with base e. Similarly, we can turn a log with another base to the natural logarithm:

  $$\log_b x = \frac{\ln x}{\ln b}$$

  That being said, there are times when other bases are useful (e.g. computer science – binary, octal, hexadecimal, or in other sciences – with their pH scales, Richter scales, and the like, based on base 10).

d. Summary

Logarithms are the inverse functions of exponential functions. Their graphs are just reflections of the graphs of exponential functions, and many properties are "reflected" as usual for inverse functions (e.g. the laws of logarithms).

If we understand the natural exponential function e^x very well, then it and its inverse function ($\ln x$) are all we really need to know about exponential functions (since any other base can be converted into base e). Historically the bases 2 and 10 have also been important: for example, earthquakes have been measured on the Richter scale, which uses $\log_{10}$ - so an earthquake of size 5 has a seismographs wave amplitude 10 times greater than an earthquake of size 4 (this corresponds to about 31 times more energy). Base 2 is very important in the computer world, since computers are based on a binary arithmatic (it's an on/off world in there!).

If you plan on continuing on in math or science, you would do well to learn these functions inside and out. They're very useful, and will come back to haunt you over and over again.

e. To do: How are the graphical elements of a logarithmic function related to

the graphical elements of the exponential function with the same base? Demonstrate with a plot.

What is the reflection of the property "the log of a product is the sum of the logs" to the exponential functions?