Calculus With Parametric Equations

MAT 229, Spring 2021

Week 14

Supporting materials

If you wish to get a different perspective on the notes below, try either of the following textbook sections.

- Stewart's *Calculus* Section 10.2: Calculus with [parametric](http://ceadserv1.nku.edu/longa/classes/2019spring/mat229/highlights/10.2.pdf) curves
- Boelkins/Austin/Schlicker's *Active [Multivariable](https://activecalculus.org/multi/) Calculus* Section 9.7: Derivatives and integrals of vector-valued functions

Review

Parametric equations are functional values of *t* equal to the *x* and *y* coordinates of a point in the plane:

 $x = f(t)$

 $y = g(t)$

(We'll soon generalize this to 3-space, adding a third coordinate -- z=h(t).)

The **parameter** in this case is *t*; and while we think of this as a variable, we also think of *x* and *y* as the variables in the problem (the coordinates of a point in the plane -- or "in space", more generally). Frequently it's useful to think of this as a motion, and of *t* as "time".

As time *t* varies over its domain, a point undergoes a corresponding motion in the plane.

Questions

Verify that the following parameterizations are all for the same curve.

- $x = 2 cos(3 t), y = 2 sin(3 t), 0 ≤ t ≤ π/3$
- $x = t$, $y = \sqrt{4 t^2}$, $-2 \le t \le 2$
- $x = 2 \sin(t)$, $y = 2 \cos(t)$, $-\pi/2 \le t \le \pi/2$

[\(Video](https://nku.zoom.us/rec/share/bO3xjHUr42nC7qaLSAoHBH61iaWBOePv6E6GDkjmRvdkgr1eaNxAmLrpZaF3BKo4.JMU3RFpU1HzhFlk7))

Tangent lines

Questions

- If $y = x^2 + 1$, compute $\frac{dy}{dx} \Big|_{x=1}$. What does this number represent?(<u>Video</u>)
- Consider the Tschirnhausen curve given by $y^2 = x^3 + 3x^2$.
	- \blacksquare Verify (1, 2) is a point on this curve.
	- \blacksquare Use implicit differentiation to find an equation for the tangent line to this curve at (1, 2).

[\(Video](https://nku.zoom.us/rec/share/hc4c9LPX1RDVZF07WHCE1qY_l8gZQ13rHQ7cgBYJTuNrgsU8My8LNwZxXaf_oSDs.f4PK0feHS2xzvFiO))

Slopes of curves defined by parametric equations

Slope is rise over run. Infinitesimally, this is $\frac{d\gamma}{dx}$. For parametric equations, the chain rule implies *dy dx* ⁼ *dy*/*dt dx*/*dt* .

Questions

- If $x = 2t^2 + 1$, $y = 3t^3 4$, find $\frac{dy}{dx}$.(<u>Video</u>)
- $x = 2(t \sin(t))$, $y = 2(1 \cos(t))$ are parametric equations for a cycloid. Find an equation for the tangent line to this curve where $t = \pi/6$. [\(Video](https://nku.zoom.us/rec/share/e9NLqFR6blEhdwaD4659W4l0GgE_ByHL5Rt-L5UP9wU7xymFOMsO5oYrpQEW1w4C.J1epC4TRh7Ai9mzs))
- \bf{v} *x* = cos(*t*), *y* = sin(2*t*) are parametric equations for a curve.
	- Find all points on this curve that have horizontal tangents.
	- Find all points on this curve that have vertical tangents.
	- \blacksquare Is (0, 0) on this curve?

[\(Video](https://nku.zoom.us/rec/share/QjpyKCah2pRVICVFJzyUdsM-Gb3KgGnqEmZlFHuVj3udEf2NZH_YOR5oqxOEJclk.KWVs0tHndhxmLoHa))

Curve length

Given parametric equations for a curve

$$
x = x(t), y = y(t), \ a \le t \le b
$$

the length of the parametric curve is

$$
\int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.
$$

Ouestion

Find the length of the curve $x = 3 \cos(t)$, $y = 3 \sin(t)$, $0 \le t \le \pi$. [\(Video](https://nku.zoom.us/rec/share/UneH_rW93WkWTJoFfWjiAxHSXHq-e6jK9cDWKKvwxYU661s0QuyhainD5TWYx1h5.y4BdD19ivLXCRYdG))

Note: Only on rare occasions like the above example, can these integrals be evaluated exactly. Because of the square root, usually these length integrals must o�en be integrated using numerical techniques.

Questions

- \blacksquare The cycloid $x = 2(t \sin(t))$, $y = 2(1 \cos(t))$ lies above the *x*-axis, but touches the axis occasionally. Determine the values of parameter *t* where it touches the *x*-axis.
- Find the length of one arch of the above cycloid.

[\(Video](https://nku.zoom.us/rec/share/ivlupV7yBMGinpzvGhf04MptXiSsvezSS0L-Zl22zTzAGVoySdfuFwT9oE55iVHI.vdv0PfIJZwn7Ikqf))

Questions

Consider the parametrically defined curve

 $x = cos(3 t)$

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y = \sin(2 t)
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- This curve starts repeating. What is its period?
- Set up the integral to find the length of this curve.

[\(Video](https://nku.zoom.us/rec/share/8FBfonIYfN8rrr1oyUH99wiQEatrjzxZAU-0gdJvRoIDBJHtFdw3rf-RERcBYp0n.vO-LxT_BqUZWoEo4))

Why the formula works

For a given curve defined by parametric equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, divide [a, b] into *n* subintervals of equal length $\Delta t = \frac{b-a}{n}$,

 $t_0 = a, t_1 = a + \Delta t, t_2 = a + 2\Delta t, t_3 = a + 3\Delta t, ..., t_n = a + n\Delta t = b$ Approximate the length of the curve using the line segments from $(x(t_i), y(t_i))$ to $(x(t_{i+1}), y(t_{i+1}))$, as *t* goes from t_0 to t_n .

Question

What is the length of the line segment from point $(x(t_i), y(t_i))$ to $(x(t_{i+1}), y(t_{i+1}))$?

An approximation for the length of the curve is the sum of the lengths of these individual line segments.

Length
$$
\approx \sqrt{((x(t_0) - x(t_1))^2 + (y(t_0) - y(t_1))^2)}
$$
 +
\n $\sqrt{((x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2)}$ + ... + $\sqrt{((x(t_{n-1}) - x(t_n))^2 + (y(t_{n-1}) - y(t_n))^2)}$
\n $= \sum_{i=1}^n \sqrt{((x(t_{i-1}) - x(t_i))^2 + (y(t_{i-1}) - y(t_i))^2)}$

The squared quantities are differences. Make them into difference quotients.

$$
\begin{split} & = \Sigma_{i=1}^n \sqrt{ \Big(\frac{(x(t_{i-1})-x(t_i))^2}{\Delta t^2} \, \Delta \, t^2 + \frac{(y(t_{i-1})-y(t_i))^2}{\Delta t^2} \, \Delta \, t^2 \Big) } \\ & = \Sigma_{i=1}^n \sqrt{ \Big(\Big(\frac{(x(t_{i-1})-x(t_i))^2}{\Delta t^2} + \frac{(y(t_{i-1})-y(t_i))^2}{\Delta t^2} \Big) \, \Delta \, t^2 \Big) } \\ & = \Sigma_{i=1}^n \, \Delta \, t \, \sqrt{ \Big(\Big(\frac{x(t_{i-1})-x(t_i)}{\Delta t} \Big)^2 + \Big(\frac{y(t_{i-1})-y(t_i)}{\Delta t} \Big)^2 \Big) } \end{split}
$$

For large *n* this becomes a better approximation. In the limit as *n* → ∞ this expression "melts" into the integral

$$
\int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.
$$

Homework

■ IMath problems on the Calculus of Parametric Equations