# Inverse Functions

### Related to Section 6.1 in Stewart, 8th edition

## Overview

We need and use a number of different kinds of functions:

- Linear functions $f(x) = m\,x + b$ are the simplest functions. We can approximate many of the simplest behavior with these “straight line” functions.
- Quadratic functions $f(x) = a\,x^2 + b\,x + c$ are useful to model things like the effects of gravity and shapes used in satellite dishes and headlights (“parabola” or “paraboloid”).
- Polynomial functions are used to approximate complicated phenomena. For instance piecewise cubic polynomials (“splines”) are used in drawing applications like Adobe Illustrator.
- Sinusoidal functions with sines and cosines model repeating (“periodic”) phenomena.

### Questions

Solve the following. (In these expressions $x$ and $y$ are called **variables**, whereas $a$ and $b$ are called **parameters**. Think of a parameter as an *unspecified yet fixed* number.)

- Solve the linear equation $y = 3\,x + 5 = b$ for $x$. (Solution)
- Solve the quadratic equation $\mathrm{y} = \frac{x^2}{2} - b = 1$ for $x$. (Solution)
- Solve the cubic equation $y = 2\,x^3 - 3 = a$ for $x$. (Solution)
- Solve the sinusoidal equation $y = 2\sin(x) = 1$ for $x$. (Solution)

What’s troubling about the

  - quadratic inverse problem?
  - sinusoidal inverse problem?

## Inverse functions

Inverse functions is a framework for solving equations. We have “obvious” methods for solving some of them, e.g.

- Use square roots to solve quadratic equations.
- Use cube roots to solve simple cubic equations.

### Definition

Given a function $f(x)$: if $y = f(x)$ has exactly one solution $x$ for any $y$ in the range of $f$, then the inverse to $f$ exists, and is denoted $f^{-1}$:

$$y = f(x) \longleftrightarrow f^{-1}(y) = x$$

We say that $f$ is "one-to-one"; we pronounce "$\longleftrightarrow$" as "if and only if".

## Example

If $f(x) = 2x^3 - 3$, then to find $f^{-1}$ solve

$y = f(x) = 2x^3 - 3$ for $x$:

$\longrightarrow y + 3 = 2x^3$

$\longrightarrow \frac{y+3}{2} = x^3$

$\longrightarrow x = \sqrt[3]{\frac{y+3}{2}}$

The inverse function to $f(x) = 2x^3 - 3$ is $f^{-1}(y) = \sqrt[3]{\frac{y+3}{2}}$ or

$f^{-1}(x) = \sqrt[3]{\frac{x+3}{2}}$

## Question

If $g(x) = \frac{x^2}{2} + 1$ find $g^{-1}(x)$, if it exists. (Solution)

---

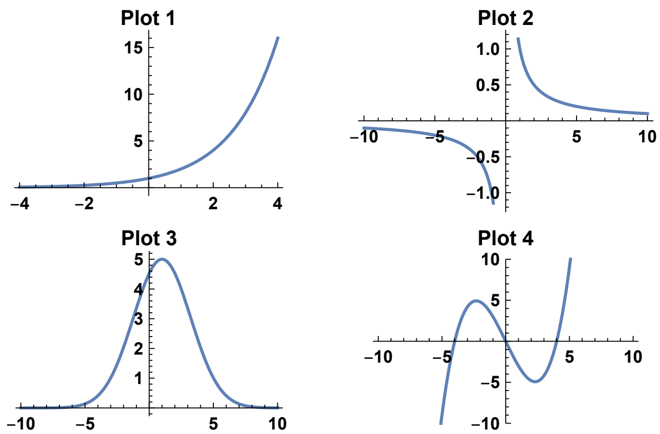
# Inverse function properties

- A function $f$ is one-to-one if it never takes on the same value twice, i.e. $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. One-to-one functions have exactly one inverse.
- A horizontal line can cross the graph of a one-to-one function at most once.

## Question

Which is the graph of a one-to-one function? If it is not a one-to-one function, how can the domain be restricted to make a one-to-one function? (Solution)

Out[∘]=

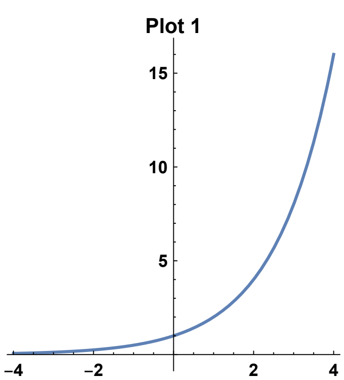



## Properties

- $f^{-1}(f(x)) = x$ for every $x$ in the domain of $f$.
- $f(f^{-1}(y)) = y$ for every $y$ in the domain of $f^{-1}$.
- The domain of $f^{-1}$ is the same as the range of $f$, and the range of $f^{-1}$ is the same as the domain of $f$.
- Since $y = f(x)$ is the same as $f^{-1}(y) = x$, the graph $y = f^{-1}(x)$ is the same as the graph $y = f(x)$ except $x$ and $y$ have been exchanged: i.e. the graph has been reflected about the line $y = x$.

## Questions

Each question refers to the function in Plot 1:




- What is the graph of the inverse function? (Solution)
- What is the domain of the inverse function? (Solution)
- Estimate the value of the inverse function $f^{-1}(x)$ when $x = 4$. (Solution)

# Calculus of inverses

Use the fact that $y = f^{-1}(x)$ is the same as $f(y) = x$, and implicit differentiation to find the derivative of $f^{-1}(x)$.

## Example

We know what the derivative of $\sqrt[3]{x}$ , but we can also get it from the cube function:

$$y = \sqrt[3]{x} \longrightarrow x = y^3$$
$$\longrightarrow \frac{d}{dx}(x) = \frac{d}{dx}(y^3)$$
$$\longrightarrow 1 = 3\,y^2\,\frac{dy}{dx}$$
$$\longrightarrow \frac{dy}{dx} = \frac{1}{3\,y^2} = \frac{1}{3\left(\sqrt[3]{x}\right)^2}$$

## Question

Apply this reasoning to find a formula for $\frac{d}{dx} f^{-1}(x)$. (Solution)

---

# Homework

- IMath problems on inverse functions