Logarithmic Functions

Supporting materials

If you wish to get a different perspective on the notes below try either of the following textbook sections.

- Stewart's *Calculus*
  Section 6.3: Logarithmic functions
- Boelkins/Austin/Schlicker's Active Calculus
  Section 2.6: Derivatives of inverse functions

Properties of the natural exponential function

The natural exponential function is the function $f(x) = e^x$ where $e \approx 2.71828\ldots$ It is "natural" in the calculus sense because it is the special exponential function that is its own derivative: $\frac{d}{dx} e^x = e^x$. That means that it's its own anti-derivative, as well (up to a constant C, of course).

The graph of $f(x) = e^x$ is below.

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Plot[E^x, {x, -2, 2}]
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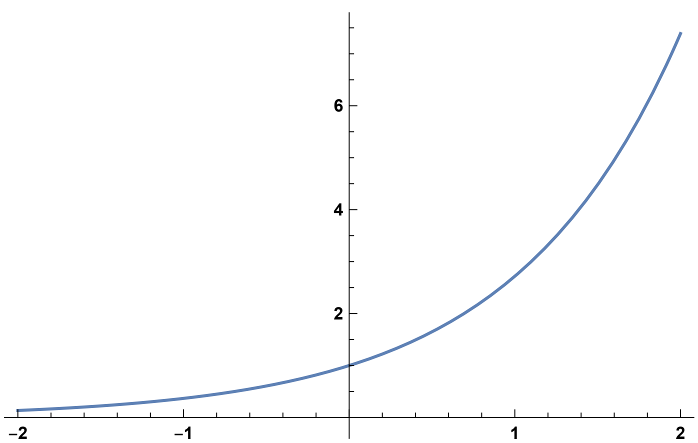

From this graph we get the following properties of $f(x)$:

- The domain of $f(x)$ is $(-\infty, \infty)$. Its range is $(0, \infty)$.
- The y-intercept of the graph is $f(0) = 1$.
- The end behavior is
  - $\lim_{x\to\infty} f(x) = \infty$
  - $\lim_{x\to-\infty} f(x) = 0$

- $f(x)$ a one-to-one function so it has an inverse function.

Questions

One mathematical model for how rumors spread is based on the function

$$p(t) = \frac{1}{1 + a\, e^{-k t}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a, k are positive constants.

- What is $\lim_{t \to \infty} p(t)$ and what does it mean for the rumor?
- Find the rate of the spread of the rumor.

Question

- The bell of a trumpet is a surface of revolution obtained by rotating an exponential curve about the horizontal axis. Find the volume of the region bounded by $y = e^x$, $x = 0$, $x = 1$, $y = 0$ rotated about the x-axis.

The natural logarithm function

Since $f(x) = e^x$ is a one-to-one function, it has an inverse function. That function is called the *natural logarithm* function and is denoted

$$f^{-1}(x) = \ln(x)$$

Everything we know about $\ln(x)$ derives from the fact that

$$y = \ln(x) \longleftrightarrow x = e^y$$

Questions

- What is the domain and range of $\ln(x)$?
- What is $\ln(1)$?
- Graph $y = \ln(x)$
- What is $\lim_{x \to \infty} \ln(x)$? What is $\lim_{x \to 0^+} \ln(x)$?
- What is the solution to $e^{2x-1} = 5$?

Properties of logarithms

In general $f(x) = a^x$ is a one-to-one function as long as base a is a positive number other than 1. Since it is one-to-one, it has an inverse and that inverse is denoted

$$f^{-1}(x) = \log_a(x)$$

This means $y = \log_a(x)$ and $x = a^y$ are equivalent.

Questions

- What is $a^{\log_a(x)}$?
- What is $\log_a(a^x)$?
- So if 2^8=256, what is $\log_2(256)$?

Properties of Logs are reflections of Exponentials

Other properties of logarithms come directly from properties of exponentials. These properties are simply “reflections” of the corresponding properties for exponentials in the “mirror” $x=y$.

- $a^x a^y = a^{x+y} \longleftrightarrow \log_a(u\, v) = \log_a(u) + \log_a(v)$
- $\frac{a^x}{a^y} = a^{x-y} \longleftrightarrow \log_a\left(\frac{u}{v}\right) = \log_a(u) - \log_a(v)$
- $(a^x)^y = a^{x\,y} \longleftrightarrow \log_a(u^r) = r\log_a(u)$

Since the natural base e is **the best for calculus**, other exponentials are rewritten in terms of e if any calculus is to be done:

$$a^x = e^{\ln(a^x)} = e^{x\ln(a)}$$

Example

Using the fact that e^x and $\ln(x)$ are inverse, $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$.

ln(2) looks scary, but it is just a number. To 20 decimal places it’s

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N[Log[2], 20]
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0.69314718055994530942
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Questions (switch to base e!):

- What is $\frac{d}{dx}2^x$?
- What is $\frac{d}{dx}5^x$? (Video)
- What is $\int e^x\,dx$?
- What is $\int 10^x\,dx$? (Video)
- Find an equation for the tangent line to $y = 5^x$ at $x = 1$.
- What is the area of the region bounded by $y = 10^x$ and the x-axis for $-1 \le x \le 1$?

Homeworks (in progress)

- IMath problems on logarithmic functions.