Elementary Functions

MAT 229, Spring 2021

Week 3

The function typically used in Calculus courses are called *elementary functions.*

Calculus I functions

Polynomial functions

Polynomial functions are those that involve only addition, subtraction, and multiplication.

Example

$f(x) = 5x^3 - 8x^2 + 3x - 15$

Facts

- Easiest functions to work with. Later in the semester we will approximate more complicated functions with polynomials.
- Domain: $(-\infty, \infty)$
- End behavior: Except for constant functions, the limit as $x \to \infty$ or $x \to -\infty$ will be $\pm\infty$.
- To differentiate use the sum and power rules: $(x^n)' = n\,x^{n-1}$.
- To integrate use the sum and power rules: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$.

Rational functions

Rational functions are those that involve addition, subtraction, multiplication, and division. They can be written as the ratio of two polynomials.

Example

$g(x) = \frac{9x^2+1}{4x-7}$

Facts

- Domain: All real numbers *except* those that cause division by zero.

- End behavior: There may be horizontal asymptotes as $x \to \pm\infty$. There may be vertical asymptotes where the function is undefined.
- To differentiate a rational function use the power rule and the quotient rule.
- Integration becomes harder. Logarithms and inverse tangents may appear in the integrals of rational functions. For example,
  - $\int \frac{1}{x}\, dx = \ln |x| + C$
  - $\int \frac{1}{1+x^2}\, dx = \tan^{-1}(x) + C$

Trigonometric functions

The six trigonometric functions are

- $\sin(x)$
- $\cos(x)$
- $\tan(x) = \frac{\sin(x)}{\cos(x)}$
- $\csc(x) = \frac{1}{\sin(x)}$
- $\sec(x) = \frac{1}{\cos(x)}$
- $\cot(x) = \frac{\cos(x)}{\sin(x)}$

Facts

- Used to model repeating (periodic) behavior.
- Domains:
  - Both sine and cosine have domains $(-\infty, \infty)$
  - The remaining four are defined for all real values except those that cause division by zero.
- The range for both sine and cosine is $[-1, 1]$.
- End behavior: There are vertical asymptotes for tangent, cosecant, secant, and cotangent where they are not defined.
- Derivatives:
  - $(\sin(x))' = \cos(x)$
  - $(\cos(x))' = -\sin(x)$
  - $(\tan(x))' = \sec^2(x)$
  - $(\csc(x))' = -\csc(x)\cot(x)$
  - $(\sec(x))' = \sec(x)\tan(x)$
  - $(\cot(x))' = -\csc^2(x)$
- Integrals:
  - $\int \sin(x)\, dx = -\cos(x) + C$

- $\int \cos(x)\,dx = \sin(x) + C$
- $\int \tan(x)\,dx$ will get to in coming weeks
- $\int \csc(x)\,dx$ will get to in coming weeks
- $\int \sec(x)\,dx$ will get to in coming weeks
- $\int \cot(x)\,dx$ will get to in coming weeks

Calculus II functions

Exponential functions

The simple exponential functions can be written as $h(x) = b^x$ where b is a positive number other than 1. General exponential functions have the form $k(x) = p(x)^{q(x)}$.

Facts

- Used to model growth and decay behavior.
- For calculus **always write simple or general functions in base *e***.
  - Simple exponential functions: $b^x = e^{r\,x}$ where $r = \ln(b)$.
  - General exponential functions: $p(x)^{q(x)} = e^{k(x)}$ where $k(x) = q(x)\ln(p(x))$.
- Domains:
  - For simple exponential functions the domain is $(-\infty, \infty)$.
  - For general exponential functions the domain is that of the exponent function $k(x) = q(x)\ln(p(x))$.
- The range for simple exponential functions is $(0, \infty)$.
- End behavior: One end, either $x \to -\infty$ or $x \to \infty$, goes to zero. The other end, $x \to \infty$ or $x \to -\infty$, goes to ∞.
- To differentiate exponential functions, use the exponential rule $(e^x)' = e^x$ along with the chain rule.
- Integration: $\int \ln(x)\,dx$ will get to in coming weeks.

Logarithm functions

Logarithms are the inverse functions to simple exponential functions.

$$y = b^x \longleftrightarrow x = \log_b(y)$$

Just as for exponential functions, we prefer to work with base e, so we **convert other logs to base *e* logs**, via

$$\log_b(y) = \frac{1}{\ln(b)}\ln(y)$$

Facts

- The domain of $\log_b(x)$ is the range of its inverse b^x,
  Domain of $\log_b(x)$: $(0, \infty)$
- The range of $\log_b(x)$ is the domain of its inverse b^x,
  Range of $\log_b(x)$: $(-\infty, \infty)$
- End behavior for natural logarithm $\ln(x)$ with domain $(0, \infty)$,
  - $\lim_{x\to 0^+} \ln(x) = -\infty$
  - $\lim_{x\to\infty} \ln(x) = \infty$
- Differentiation: $(\ln(x))' = \frac{1}{x}$
- Integration: $\int \ln(x)\, dx$ we will get to in coming weeks.

Inverse trigonometric functions

These are the inverse functions to the trigonometric functions. However, the domains must be restricted.

- $y = \sin(x),\ -\frac{\pi}{2} \le x \le \frac{\pi}{2} \longleftrightarrow x = \sin^{-1}(y)$
- $y = \tan(x),\ -\frac{\pi}{2} < x < \frac{\pi}{2} \longleftrightarrow x = \tan^{-1}(y)$
- $y = \cos(x),\ 0 \le x \le \pi \longleftrightarrow x = \cos^{-1}(y)$

Facts

- The domains of each is the range of its inverse.
  - Domain $\sin^{-1}(x)$ is $[-1, 1]$.
  - Domain $\cos^{-1}(x)$ is $[-1, 1]$.
  - Domain $\tan^{-1}(x)$ is $(-\infty, \infty)$.
- The range of each is the restricted domain of its inverse
  - Range $\sin^{-1}(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
  - Range $\cos^{-1}(x)$ is $[0, \pi]$.
  - Range $\tan^{-1}(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- $\tan^{-1}(x)$ is the only one of these three with an unbounded domain.
  - $\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$
  - $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$
- Differentiation:
  - $(\sin^{-1}(x))' = \frac{1}{\sqrt{1-x^2}}$

- $\left(\cos^{-1}(x)\right)' = -\frac{1}{\sqrt{1-x^2}}$
- $\left(\tan^{-1}(x)\right)' = \frac{1}{1+x^2}$

- Integration: we will get to it in the coming weeks.

Combinations

All of the above functions can be combined in different ways using addition, subtraction, multiplication, division, and composition.

Facts

- Differentiation: use the product rule, the quotient rule, and the chain rule along with the above differentiation rules.
- Limits: one thing we've done recently is slip limits inside of compositions -- which we can do if the functions are continuous.
- Integration:
  - substitution can be used in some cases
  - we will develop other integration techniques
  - there are functions that don't have "nice" integrals, no techniques exist to write their antiderivatives easily. For these we require numerical techniques to approximate the integral values.

Questions

- Let f represent the general exponential function $f(x) = x^x$.
  - How would it be written base e?
  - What is its domain? (Answer only after you switch to base e.)
  - What are its critical points? For each determine if it is a maximum, a minimum, or neither.

  (Video)
- Let g represent the combined function $g(x) = e^{-x} \tan^{-1}(x)$.
  - What is its domain?
  - Describe its end behavior.
  - What is its linear approximation at $x = 0$?

  (Video)