§3.3 #39

Given m divisions are required to find the $\gcd(a,b)$, $m \geq 4$.

Prove: $m < \log_{1.5} a - 1$ $(m \geq 4)$

Assume #38: $\left(\frac{3}{2}\right)^{m+1} < F_{m+2} \leq a$ $m \geq 4$

$$\left(\tfrac{3}{2}\right)^{m+1} < a$$

$\log_{3/2}(x)$ is growing, & preserves order:

If $x < y$, then

$$\log_{3/2}(x) < \log_{3/2}(y)$$

$$\therefore \log_{3/2}\left(\tfrac{3}{2}\right)^{m+1} < \log_{3/2}(a)$$

$$\therefore m+1 < \log_{3/2}(a)$$

$$\therefore \boxed{m < \log_{3/2}(a) - 1}$$ ✓

or $$m < \frac{\log_2(a)}{\log_2(3/2)} - 1$$

Turns out that $\dfrac{1}{\log_2(3/2)} \approx 1.71$

$$\boxed{m < 1.72 \log_2(a) - 1}$$ *

Better than our author's sloppy $m \leq 2\log_2 a$

#40. a $89 = 1\cdot 55 + 34$

$55 = 1\cdot 34 + 21$

$34 = 1\cdot 21 + 13$

$21 = 1\cdot 13 + 8$

$13 = 1\cdot 8 + 5$

$8 = 1\cdot 5 + 3$

$5 = 1\cdot 3 + 2$

$3 = 1\cdot 2 + \boxed{1}$

$2 = 2\cdot 1 + 0$

gcd(89, 55) required 9 divisions.

b. $E(89) \leq 2\log_2(89) \approx 12.95$ ✓

c. $E(89) < 1.72\cdot\log_2(89) - 1 \approx 10.07$ ✓

d. $E(89) < 5\cdot 2 = 10$ ✓

Lamé's Theorem:

$$E(n) \leq 5\lceil \log_{10}(n)\rceil$$
$$\leq 5\left\lceil \frac{\log_2(n)}{\log_2(10)}\right\rceil$$
$$\leq 5\left(\frac{\log_2(n)}{\log_2(10)} + 1\right)$$
$$\leq \underbrace{\frac{5}{\log_2(10)}}\log_2(n) + 5$$
$$1.50\log_2(n) + 5$$

$1.45 < \underbrace{1.50}_{\text{tighter yet!}} < \underbrace{1.72}_{\text{tighter}} < \underbrace{2.00}_{\text{sloppy}}$

Last time:

$$m \approx \frac{\log_2(n)}{\log_2(8)} + \underbrace{\frac{\log_2(\sqrt{5})}{\log_2(8)} - 2}_{\approx 1.34}$$

$$E(n) = m < 1.45\log_2(n) + 1.35$$

$$\frac{\log(13)}{\log(8)} + \frac{\log 15}{\log(8)} \cdot 2$$

Fibonacci Nim (only don't call it that - it gives away the strategy!

- Two player game, with an n counters (e.g. chocolate eggs)
- Objective: winner is the player who picks up the last counter.
- First player: takes m counters, where $1 \leq m \leq n-1$ (can't pass; can't take them all).
- Player 2: takes anywhere from 1 to $2m$ counters (can't pass)
- Play continues with the "1 to twice what the previous player took" rule until the final counter is taken.

Go 1st if n is not Fibonacci ✓

$\boxed{26}$ counters - do I go 1st or 2nd?

Go first.

$$\begin{array}{r} 26 \\ -\ 5 \\ \hline 21 \\ -\ 3 \\ \hline 18 \\ -\ 5 \end{array}$$

26 = 21 + $\boxed{5}$

Sum of two non-consecutive fibonacci numbers

$$\begin{array}{r} 13 \\ -\ 2 \\ \hline 11 \\ -\ 3 \\ \hline 8 \\ -\ 2 \\ \hline 6 \\ -\ 1 \\ \hline 5 \\ -\ 1 \\ \hline 4 \\ -\ 1 \\ \hline 3 \\ -\ 1 \\ \hline \boxed{2} \end{array}$$

You win!

18 → 13, 5

11 → 8, $\boxed{3}$

6 → 5, $\boxed{1}$

4 → 3, $\boxed{1}$

Go second if n is Fibonacci.