# Section 4.1: Sets

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**Abstract**

This section, the only section we consider from Chapter 4, simply gives us some basic vocabulary and notions of sets that we will need when we get to Boolean algebras later. We also make an observation about the rules satisfied by the binary operations of "intersection" and "union" from set theory, and their connection to the rules of the binary connectives $\wedge$ and $\vee$ of propositional logic.

One of the most important ideas is that of the "power set" – the set of all subsets of a set.

It also includes some really interesting examples of ideas from set theory (e.g. different sizes of infinite sets – did you know that infinity comes in infinitely many different sizes?).

## 1 Notation

A set (call it $A$) is loosely a collection of objects, within some universe; the objects are called the **elements** of $A$.

Capital letters denote sets, and $\in$ denotes inclusion in a set, so that $x \in A$ means that $x$ is a member (or element) of a set, and $x \notin A$ means that $x$ isn't a member.

Sets are unordered: the order in which the elements are listed is unimportant.

We can use predicate logic to determine (or even define) when two sets are **equal**:

$$A = B \iff (\forall x)[(x \in A \to x \in B) \wedge (x \in B \to x \in A)]$$

The notation for a set whose elements are characterized by possessing property $P$ is

$$S = \{x | P(x)\}$$

| = "such that"

and is read "$S$ is the set of all $x$ such that $P(x)$"

One curiously useful set is the **empty** set, denoted $\emptyset$ or $\{\}$.

Some important sets of numbers:

| | |
|---|---|
| $\mathbb{N}$ | Natural numbers – although our author throws in 0, argh! |
| $\mathbb{Z}$ | Integers – positive and negative natural numbers, plus 0 |
| $\mathbb{Q}$ | Rational numbers – reals expressible as ratios of integers |
| $\mathbb{I}$ | Irrational numbers – reals **not** expressible as integers |
| $\mathbb{R}$ | Real numbers – the continuum of numbers on the number line |
| $\mathbb{C}$ | Complex numbers – including the important $i = \sqrt{-1}$ |

**Example**: Practice 3, p. 224. Describe each set:

(a) $A = \{x | x \in \mathbb{N} \wedge (\forall y)(y \in \{2, 3, 4, 5\} \rightarrow x \geq y)\}$

(b) $B = \{x | (\exists y)(\exists z)(y \in \{1, 2\} \wedge z \in \{2, 3\} \wedge x = y + z)\}$

$= \{3, 4, 5\}$

## 2 Relationships between Sets

$A$ is a **subset** of $B$, denoted $A \subseteq B$, if

$$(\forall x)(x \in A \rightarrow x \in B)$$

and $A$ is a **proper subset** of $B$, denoted $A \subset B$, if

$$(\forall x)(x \in A \rightarrow x \in B) \wedge (\exists x)(x \notin A \wedge x \in B)$$

**Example**: Practice 6, p. 225

$A = \{x \mid x \in \mathbb{N} \wedge x \geq 5\}$

$B = \{10, 12, 16, 20\}$

$C = \{x \mid (\exists y)(y \in \mathbb{N} \wedge x = 2y)\}$

Which are true?

a. $B \subseteq C$
b. $B \subset A$
c. $A \subseteq C$
d. $26 \in C$
e. $\{11, 12, 13\} \subseteq A$
f. $\{11, 12, 13\} \subset C$
g. $\{12\} \in B$
h. $\{12\} \subseteq B$
i. $\{x \mid x \in \mathbb{N} \wedge x < 20\} \not\subseteq B$
j. $5 \subseteq A$
k. $\{\emptyset\} \subseteq B$ ($\emptyset \subseteq B$ true)
l. $\emptyset \notin A$

Theorem:

$$A = B \iff A \subseteq B \wedge B \subseteq A$$

## 3 Sets of Sets

**Power Set**: Given set $S$, the power set of $S$, denoted $\wp(S)$, is the set of all subsets of $S$. (Note that $S$ and $\emptyset$ are themselves elements of the power set of $S$.)

**Example**: How big is the power set of a given set? (Practice 8 and 9, p. 227) $A = \{1, 2, 3\}$

Pascal's triangle gives you the breakdown on the number of various sized subsets you can create from a set of a given size. The line number $n$ in the triangle (indexed from 0) tells you the size ($n$) of the underlying set, and the total across the row tells you just how many subsets there are ($2^n$). This is called the **cardinality** of the set: $Card(\wp(S)) = 2^n$ – its size. But cardinality is used even for infinite sets.

(1) $A \subseteq A$   (1) $\emptyset \subseteq A$

(3) $\{1\} \subseteq A$, $\{2\} \subseteq A$, $\{3\} \subseteq A$

(3) $\{1,2\} \subseteq A$, $\{1,3\} \subseteq A$, $\{2,3\} \subseteq A$

There are $8 = 2^3$

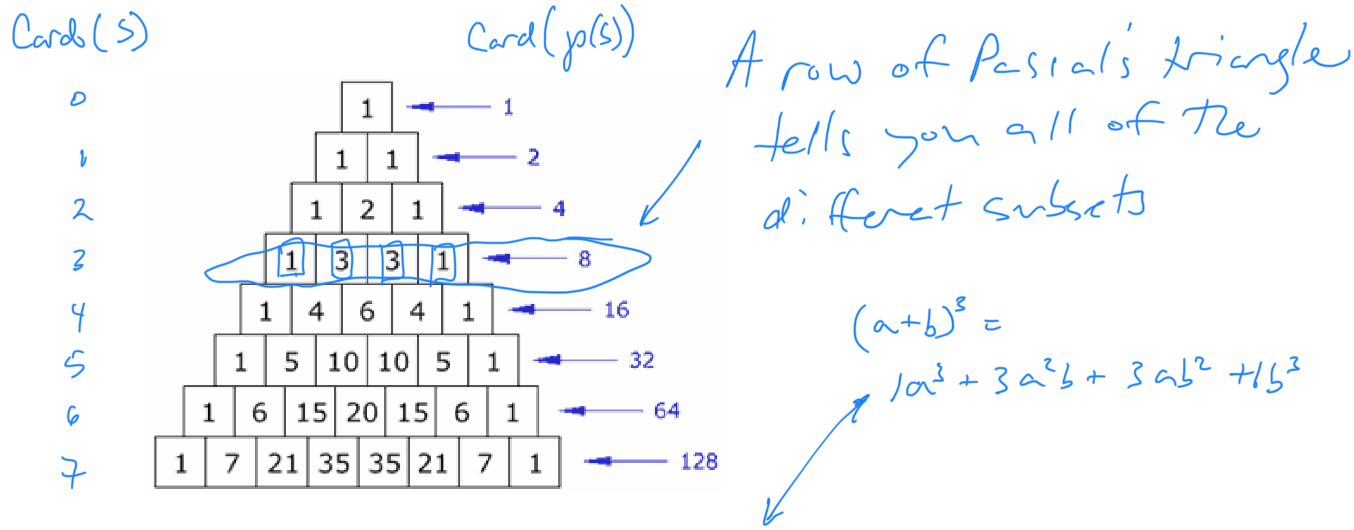


Figure 1: Pascal's Triangle (aka binomial coefficients)

## 4 Binary and Unary Operations

We can create **ordered pairs** of elements of a set. From $A = \{1, 3, 4\}$ we can create the ordered pairs $(1, 3)$ and $(3, 3)$, for example. Now the order of the elements is important!

**Question:** How many distinctly different ordered pairs are there if we have a set with $n$ elements?

**Definition**: $\circ$ is a **binary operation** on a set $S$ if for every ordered pair $(x, y)$ of elements of $S$, $x \circ y$ exists, is unique, and is a member of $S$.

**Definition**: $\circ$ is **well-defined** if $x \circ y$ exists and is unique.

**Definition**: $\circ$ is **closed** if $x \circ y \in S$.

Three ways to fail to be a binary operation on $S$:

(a) there are pairs for which $x \circ y$ fails to exist;

(b) there are pairs for which $x \circ y$ gives multiple different results;

(c) there are pairs for which $x \circ y$ doesn't belong to $S$.

**Definition**: a **unary operation** on a set $S$ associates with every element $x$ of $S$ a unique element of $S$.

**Example**: Practice 12, p. 230

# 5 Operations on Sets

Given a set $S$ of elements of interest (the **universal set**), we may want to operate on various subsets of $S$ (that is, elements of $\wp(S)$). For example,

**Definition**: Let $A, B \in \wp(S)$. The **union** of $A$ and $B$, denoted $A \cup B$, is given by $\{x|x \in A \vee x \in B\}$. The **intersection** of $A$ and $B$, denoted $A \cap B$, is given by $\{x|x \in A \wedge x \in B\}$.

These are examples of binary operations on the set of power sets of a set.

**Definition**: For a set $A \in \wp(S)$, the **complement** of $A$, denoted $A'$, is $\{x|x \in S \wedge x \notin A\}$.

**Definition**: For set $A, B \in \wp(S)$, the **set-difference** of $A$ and $B$, denoted $A - B$, is given by $\{x|x \in A \wedge x \notin B\}$.

Venn Diagrams are useful tools for considering the notions of union and intersection. The diagrams in Figures 4.1 and 4.2 (p. 231) illustrate these notions "pictorially".

**Examples**:

(a) Practice 14, p. 232: illustrate $A'$ using a Venn Diagram.

(b) Practice 15, p. 232: illustrate $A - B$ using a Venn Diagram.

**Definition**: For set $A, B \in \wp(S)$, the **Cartesian product (cross product)** of $A$ and $B$, denoted $A \times B$, is the set of all ordered pairs, and is given by

$$A \times B = \{(x, y)|x \in A \wedge y \in B\}.$$

# 6 Set Identities

We will encounter the following "Set identities" later in the context of "Boolean algebras":

| | | |
|---|---|---|
| 1$a$. $A \cup B = B \cup A$ | 1$b$. $A \cap B = B \cap A$ | commutative property |
| 2$a$. $(A \cup B) \cup C = A \cup (B \cup C)$ | 2$b$. $(A \cap B) \cap C = A \cap (B \cap C)$ | associative property |
| 3$a$. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | 3$b$. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | distributive property |
| 4$a$. $A \cup \oslash = A$ | 4$b$. $A \cap S = A$ | identity property |
| 5$a$. $A \cup A' = S$ | 5$b$. $A \cap A' = \oslash$ | complement property |

Notice the "dual" nature of the properties: it seems that the operations of $\cup$ and $\cap$ have a lot in common!

**Question**: What correspondence do you observe between these identities and those of wffs with the logical connective $\wedge$ and $\vee$?

# 7 Countable and Uncountable Sets

As an interesting application of set theory, we will now demonstrate that there are infinitely many sizes of infinity!

The natural numbers comprise the smallest infinity, a **denumerable** or **countable** infinity.

We prove that two sets are of equal size (even if infinite!) by creating a **one-to-one correspondence** between the two sets. If such a correspondence exists, then the two sets have the same size.

**Example:** The even natural numbers are the same size as the natural numbers, as shown by the one-to-one correspondence

$$f : \mathbb{N} \to \mathbb{N} \text{ given by } n \longleftrightarrow 2n$$

**Theorem**: the rational numbers are countable.

**Theorem**: the real numbers are **not** countable. (Cantor's diagonalization argument, p. 238)

**Theorem**: the power set of a set $S$ is always larger than $S$ (punch line: there is always a bigger infinity than the one you already have).

**Proof**: By contradiction. Consider $f : S \longrightarrow \wp(S)$ a one-to-one correspondence between $S$ and $\wp(S)$. That is, every set of $\wp(S)$ is represented by an element of $S$. Consider the subset of $S$ given by

$$A = \{x \in S | x \notin f(x)\}$$

If we define $f(S)$ as $f(S) \equiv \{f(x)|x \in S\}$, then the one-to-one mapping asserts that $f(S) = \wp(S)$. But $A \notin f(S)$ (because it's different from every element $f(x)$ of $f(S)$), and yet $A \in \wp(S)$. But this is a contradiction: we asserted that the mapping was one-to-one – i.e., that $f(S) = \wp(S)$.

Here's the purported mapping by $f$:

$$\begin{array}{ccl} x_1 & \to & B_1 = f(x_1) \\ x & \to & B = f(x) \\ x^* & \to & B^* = f(x^*) \\ \vdots & & \vdots \end{array}$$

But $A = \{x \in S | x \notin f(x)\}$ is different from each of the sets on the right-hand side, by construction: for example, if $x_1 \in B_1$, then $A$ rejects it; if $x^* \notin B$, then we take $x^* \in A$; and so on. It's the same argument as Cantor's diagonalization argument (Example 23, p. 238), on steroids....