§3.3 Homework - Graded 14 & 38

#14. We're looking at Bubblesort, &, if counting comparisons only in the author's implementation, we see that

$$C(n) = \underbrace{C(n-1)}_{\text{we call bubblesort with list shorter by 1}} + \underbrace{n-1}_{\text{we make } n-1 \text{ comparisons } (i=1,\dots,j-1) \text{ in the code}}$$

$C(1) = 0$ (done - write out list)

$\therefore$ $C(2) = 0 + 1$

$C(3) = C(2) + 2 = 0 + 1 + 2$

$C(4) = C(3) + 3 = 0 + 1 + 2 + 3$

$$C(n) = \underbrace{\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}}$$

We've proven this before - no need to prove it again!

Now we add in exchanges, which only occur if adjacent list elements are in the wrong order.

a. Worst case - reversed list.

We'll bubble sort the largest

5 4 3 2 1 $\longrightarrow$

$\boxed{4\ 3\ 2\ 1}\ 5$ & do it again, with 4 exchanges.

Hence

$$E(n) = E(n-1) + n - 1$$

$$E(1) = 0$$

Same recurrence! :)

Total: $C(n) + E(n) =$

$$2\left[\frac{(n-1)n}{2}\right] = \boxed{(n-1)n}$$

b. Best case: already ordered.

0 exchanges.

Total: $C(n) = \dfrac{(n-1)n}{2}$

c. "Average case": average number of exchanges $\rightarrow$ $E(n) = \dfrac{(n-1)n}{2\cdot 2}$

$$\text{Total} = \frac{(n-1)n}{2} + \frac{(n-1)n}{4}$$
$$= \boxed{\tfrac{3}{4}\, n(n-1)}$$

#38 This is a piece of cake, because we're allowed to use #37 & #26 of section 3.1.

#37 says that for m exchanges,

$$a \geq F(m+2).$$

#26 say that, for $n \geq 6$
$$F(n) > \left(\tfrac{3}{2}\right)^{n-1}$$

Hence, for $n = m+2 \geq 6$

$$F(m+2) > \left(\tfrac{3}{2}\right)^{m+2-1} \quad m \geq 4$$

Putting the two together,

$$a \geq F(n+2) > (3/2)^{n+1} \quad n \geq 4.$$

Q.E.D.

Some of you didn't think I'd let you get away with citing #26, so you decided to prove it. So let me do that one.

For $n \geq 6$ $\quad F(n) > \left(\frac{3}{2}\right)^{n-1}$ (this is $P(n)$)

need two base cases in what follows

$$\left\{ \begin{array}{l} F(6) = 8 > \left(\frac{3}{2}\right)^5 \approx 7.6 \\ F(7) = 13 > \left(\frac{3}{2}\right)^6 \approx 11.4 \end{array} \right. \checkmark$$

haven't checked yet, but virtually sure!

Assume $P(r)$ for $6 \leq r \leq k$, $k \geq 8$. Consider $P(k+1)$ & in particular its LHS:

$$F(k+1) = F(k) + F(k-1) \quad \text{(by defn. of } F\text{)}$$

$$\geq \left(\frac{3}{2}\right)^{k-1} + \left(\frac{3}{2}\right)^{k-2}$$ (by assumption of the two preceding cases)

$$= \left(\frac{3}{2}\right)^{k-2}\left(\frac{3}{2}+1\right) > \underbrace{\left(\frac{3}{2}\right)^{k-2}\left(\frac{3}{2}\right)^{2}}$$

since

$$\left(\frac{3}{2}+1\right) = \frac{5}{2} > \frac{9}{4} = \left(\frac{3}{2}\right)^2$$

Why we need to check $P(6)$ & $P(7)$

$$\therefore F(k+1) > \left(\frac{3}{2}\right)^{k-2+2} = \left(\frac{3}{2}\right)^{k}$$

$$F(k+1) > \left(\frac{3}{2}\right)^{(k+1)-1} \quad \checkmark$$

$\therefore P(k+1)$.

$\therefore P(n) \quad \forall n \geq 6$