(1) * Prove by 2nd induction

* $m = 6 \Rightarrow \begin{cases} F(m) = 8 \\ \left(\frac{3}{2}\right)^{m-1} = 7.59375 \end{cases}$ (True)

$m = 7 \Rightarrow \begin{cases} F(m) = 13 \\ \left(\frac{3}{2}\right)^{m-1} = 11.390625 \end{cases}$ (True)

* From induction, need to prove: $F(k+1) > \left(\frac{3}{2}\right)^k$

use 2nd principle to assert these

+ $F(k) > \left(\frac{3}{2}\right)^{k-1}$
+ $F(k-1) > \left(\frac{3}{2}\right)^{k-2}$

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$$\Rightarrow F(k+1) = F(k) + F(k-1)$$
$$> \left(\frac{3}{2}\right)^{k-1} + \left(\frac{3}{2}\right)^{k-2}$$
$$= \left(\frac{3}{2}\right)^{k-2}\left(\frac{3}{2} + 1\right)$$
$$= \left(\frac{3}{2}\right)^{k-2} \cdot \frac{5}{2}$$
$$> \left(\frac{3}{2}\right)^{k-2} \cdot \left(\frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^k$$

okay

$\Rightarrow F(k+1) > \left(\frac{3}{2}\right)^k$ (QED)

$\therefore F(m) > \left(\frac{3}{2}\right)^{m-1}$ for $m \geq 6$ by 2nd induction

Well done!

2) a) $V(n) = V(n-1) + V(n-2)$ where $V(1) = 2$ and $V(2) = 3$

b) On a given row, all nodes split at least once to ~~the~~ make the next row so $V(n-1)$. T nodes split twice. All nodes from the previous row split into a T node so $V(n-2)$. Adding them together, that makes $V(n) = V(n-1) + V(n-2)$

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Good

c) This is the fibonacci sequence except $f(n+2) = V(n)$

③ a) i. $S(n) = 2S(n-1)$ for $n \geq 2$

$S(1) = 1$

- We're cutting each line into two lines. So we take however many lines we had and double them.

Yes!

ii. $L(n) = \frac{1}{3}S(n-1)$ for $n \geq 2$

$L(1) = 1$

We're seperating each line into 2 lines that are 1/3 the original's length. So we take the last line's length and third it.

$2(n-2)$

Good

b) $E(n): \quad E(n-1) + 2^{n-2}\left(\frac{1}{3^{n-1}}\right)$

$c = 1 \qquad g(n) = 2^{n-2}\left(\frac{1}{3^{n-1}}\right)$

$1(0) + \sum_{i=2}^{n} 1\left(2^{n-2}\left(\frac{1}{3^{n-1}}\right)\right)$

$\sum_{i=2}^{n} 2^{n-2} \cdot \left(\frac{1}{3^{n-1}}\right)$

b. $E(1) = 0$

$\boxed{E(n) = E(n-1) + \frac{1 - E(n-1)}{3} \quad (n \geq 2)}$

Good!

Problem 3:

a) i. $S(1) = 1$

Each time n increase, for each line segments there are two more line segment

$\Rightarrow S(2) = 2; \quad S(3) = 4; \ldots; S(n) = S(n-1) \cdot 2 = 2^{n-1}$

ii. $L(1) = 1$

Each time n increase, lenght of each segment is ~~dec~~ decreased by $\frac{1}{2}$ $\frac{1}{3}$

$\Rightarrow L(2) = \frac{1}{3}; \quad L(3) = \frac{1}{9}; \ldots; L(n) = L(n-1) \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^{n-1}$

b) Total lenght of empty space is 1 minus all the segments

$$\Rightarrow E(n) = 1 - L(n) \cdot S(n)$$
$$= 1 - \frac{1}{3} \cdot L(n-1) \cdot 2S(n-1)$$
$$= 1 - \frac{1}{3} \cdot \frac{1}{3} \cdot 2 \cdot 2 \cdot L(n-2) \cdot S(n-2)$$
$$\ldots = 1 - \left(\frac{2}{3}\right)^{n-1} \cdot L(n-(n-1)) \cdot S(n-(n-1))$$
$$= 1 - \left(\frac{2}{3}\right)^{n-1} \cdot L(1) \cdot S(1)$$
$$= 1 - \left(\frac{2}{3}\right)^{n-1}$$

Base: $n = 1 \Rightarrow E(n) = 1 - \left(\frac{2}{3}\right)^{1-1} = 0$

Assume for $k \in \mathbb{N}$; we got $E(k) = 1 - \left(\frac{2}{3}\right)^{k-1}$

$$\Rightarrow E(k+1) = 1 - L(k+1) \cdot S(k+1)$$
$$= 1 - 2^k \cdot \left(\frac{1}{3}\right)^k = 1 - \left(\frac{2}{3}\right)^k = 1 - \left(\frac{2}{3}\right)^{(k+1)-1}$$

$\Rightarrow E(n) = 1 - \left(\frac{2}{3}\right)^{n-1} \quad \forall n \geq 1.$

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But you didn't write a recurrence relation for E

(4)

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a. gcd (144, 78)

$144 = 78 \cdot 1 + 66$

$78 = 66 \cdot 1 + 12$

$66 = 12 \cdot 5 + 6$

$12 = \underline{6} \cdot 2 + \underline{0}$

$\Rightarrow$ The gcd of 144 and 78 is 6

b gcd(366, 144)

$$\left.\begin{array}{l} 366 = 144 \cdot 2 + 78 \\ 144 = 78 \cdot 1 + 66 \\ \quad \vdots \\ 12 = 6 \cdot 2 + 0 \end{array}\right\} \text{gcd}(144, 78)$$

$\Rightarrow$ It ~~too~~ takes gcd(366, 144) one division to be similare to the divisions of gcd(144,78)

C. It takes 4 divisions to perform gcd (144,78)

d. Smallest integer pair is gcd (8, 5)
($(4+2)^{th}$ and $(4+1)^{th}$ of Fibonacci.