MAT385 Test 2 (Spring, 2021): 2.2, 3.1, 3.2, 3.3

Name:

Directions:

a. There are four equally weighted problems (one per page).

b. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning).

c. Indicate clearly your answer to each problem (e.g., put a box around it).

d. **Good luck!**

Problem 1: (10 pts) Let $F(n)$ denote the n^{th} Fibonacci number (with $F(1) = F(2) = 1$). Use induction to prove that $F(n) > \left(\frac{3}{2}\right)^{n-1}$ for $n \geq 6$.

$P(n): F(n) > \left(\frac{3}{2}\right)^{n-1}$

1	2	3	4	5	6	7	8
1	1	2	3	5	8	13	21

$\left(\frac{3}{2}\right)^5 = \frac{243}{32} < 8 = F(6)$

$\left(\frac{3}{2}\right)^6 = \frac{3}{2}\left(\frac{3}{2}\right)^5 < 12 < 13 = F(7)$

Two base cases ✓

We'll prove this using the 2nd principle (Because of the Fibonacci's)

Assume $P(r)$ $\forall r \in \{6, \dots, k\}$. Consider $P(k+1)$ & the LHS:

$F(k+1) = F(k) + F(k-1)$

$> \left(\frac{3}{2}\right)^{k-1} + \left(\frac{3}{2}\right)^{(k-1)-1}$

$= \left(\frac{3}{2}\right)^{k-2}\left[\frac{3}{2} + 1\right]$

$= \left(\frac{3}{2}\right)^{k-2}\left(\frac{5}{2}\right) = \left(\frac{3}{2}\right)^{k-2}\left(\frac{10}{4}\right) > \left(\frac{3}{2}\right)^{k-2}\left(\frac{9}{4}\right) = \left(\frac{3}{2}\right)^{k-2}\left(\frac{3}{2}\right)^2$

$\therefore F(k+1) > \left(\frac{3}{2}\right)^{k-2+2} = \left(\frac{3}{2}\right)^k$

$\therefore P(k+1)$ ✓

$\therefore P(n)$ $\forall n \geq 6$ by the 2nd principle of mathematical induction.

Problem 2: (10 pts) A coin is tossed n times. This figure illustrates only those tosses which never include two heads in a row ("valid tosses" – e.g. TTTHT is a valid toss when one tosses $n = 5$ times):

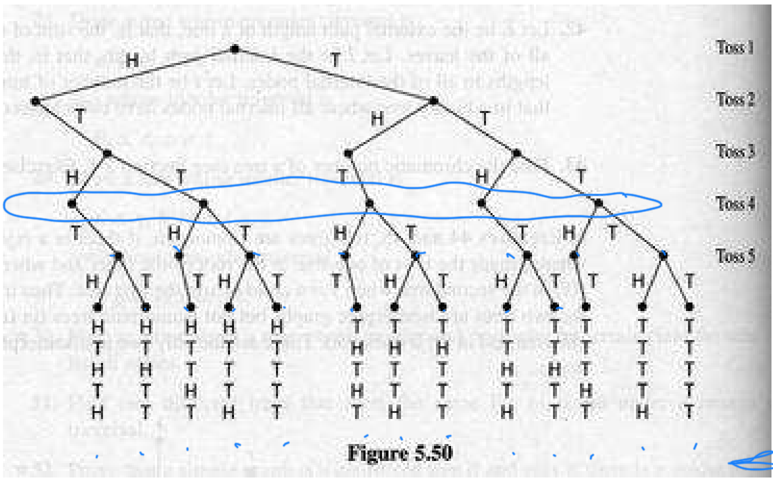


Figure 5.50

$V(1) = 2$

$V(2) = 3$ HT TH TT

$V(3) = 5$

$V(4) = 8$

$V(5) = 13 = F(7)$

a. (4 pts) **Write a recurrence relation** for the **number of valid tosses** $V(n)$ **in** n **throws**. $V(1) = 2$: H or T (see the figure).

$$V(n) = V(n-1) + V(n-2)$$

b. (3 pts) Explain/Justify your recurrence relation. How do you derive it?

c. (3 pts) Relate $V(n)$ to a familiar sequence or sequences studied in our course.

b. Those that end in T are derived from all the toss strings at $n-1$; all that end in H end in TH, & are derived from all those strings at $n-2$.

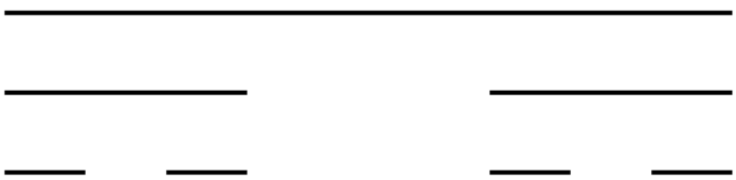
c. $V(n) = F(n+2)$

Problem 3: (10 pts) Perhaps you know the "Cantor Middle Third" fractal: not the prettiest fractal, but easily described. Fractals are often created recursively: you start with an object, subdivide it into similar objects, and then "do it again" on the resulting object(s). In this case the rules are:

a. Given a line segment, remove its middle third;

b. Recurse, on the left and right line segments.

The original segment and first two iterations are illustrated in the following figure:



$S(1) = 1 \quad L(1) = 1 \quad E(1) = 0$

$S(2) = 2 \quad L(2) = 1/3 \quad E(2) = 1/3$

$S(3) = 4 \quad L(3) = 1/9 \quad E(3) = \frac{1}{3} + 2(1/9)$

a. For the following two sequences, write a recurrence relation and give a closed-form solution (you don't need to prove it):

   i. (2.5 pts) $S(n)$, the number of line segments at step n ($S(1) = 1$).

   ii. (2.5 pts) $L(n)$, the length of each line segment at step n (consider the initial line segment to be of length one – $L(1) = 1$).

b. (5 pts) Now, write a recurrence relation for $E(n)$, the total length of **empty space** at step n ($E(1) = 0$), and solve the recurrence relation to obtain a closed form solution for $E(n)$.

a. i. $S(n) = 2S(n-1) \;;\; S(1) = 1$

$\therefore S(n) = 2^{n-1} \qquad n \geq 1$

ii. $L(n) = \frac{1}{3} L(n-1) \;;\; L(1) = 1$

$\therefore L(n) = \frac{1}{3}^{n-1} \qquad n \geq 1$

b. $E(n) = E(n-1) + \frac{1}{3}(1 - E(n-1))$

$= \frac{1}{3} + \frac{2}{3} E(n-1)$

$E(n) = c^{n-1} \cdot 0 + \sum_{i=2}^{n} c^{n-i} (\frac{1}{3})$

$= \frac{1}{3} \sum_{i=2}^{n} (\frac{2}{3})^{n-i}$

$= \frac{1}{3} \left[1 + \frac{2}{3} + \dots + (\frac{2}{3})^{n-2} \right]$

$= \frac{1}{3} \frac{1 - (\frac{2}{3})^{n-1}}{1 - \frac{2}{3}} = \boxed{1 - (\frac{2}{3})^{n-1}}$

check:

$= 1 - S(n) \cdot L(n)$

$= 1 - 2^{n-1} \cdot (\frac{1}{3})^{n-1}$

$= 1 - (\frac{2}{3})^{n-1}$

✓

Problem 4: (10 pts) The Euclidean Algorithm

a. Demonstrate the Euclidean algorithm by computing the gcd of 144 and 78.

b. What is the relationship between gcd(144,78) and gcd(366,144)?

c. How many **divisions** does it take to perform gcd(144,78)?

d. What is the smallest integer pair (a, b) to require that many divisions in gcd(a,b)?

a. gcd(144,78)

$144 = 1 \cdot 78 + 66$

$78 = 1 \cdot 66 + 12$

$66 = 5 \cdot 12 + \boxed{6}$

$12 = 2 \cdot 6 + 0$

gcd(144,78) = 6

b. gcd(366,144)

$366 = 2 \cdot 144 + 78$

$144 = 1 \cdot 78 + 66$

They're equal! gcd(366,144) takes one extra step.

c. 4 divisions

d. Consecutive Fibonacci #s:

$2 = 2 \cdot 1 + 0$

$3 = 1 \cdot 2 + 1$

$5 = 1 \cdot 3 + 2$

$8 = 1 \cdot 5 + 3$

gcd(8,5) requires 4.