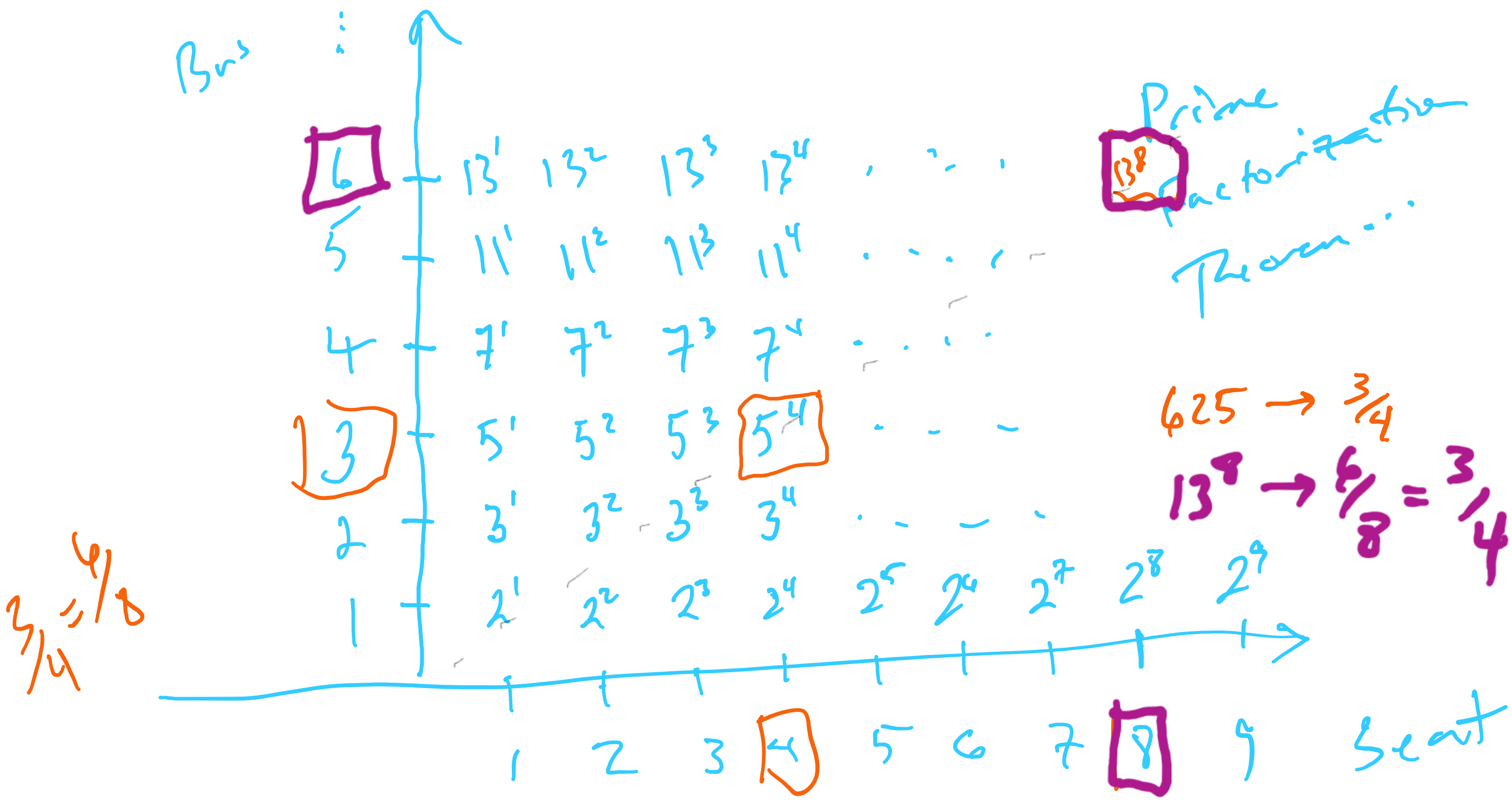

$\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ $\qquad$ $\mathbb{N} \to \mathbb{Q}^+$

A proper subset of an infinite set can be the same size as the set itself, but it can never be bigger!

① $\mathcal{P}(S)$ is at least as big as S.

$\forall x \in S \qquad \{x\} \in \mathcal{P}(S)$

So $\mathcal{P}(S)$ can "swarm" S, with $\{x\} \to x$

(+ there are lots of other elements of

$\mathcal{P}(S)$ to map onto elements of S — e.g. φ.

(2) Assume S has the same cardinality as $\mathcal{P}(S)$ (they're the same size): so $\exists$ a one-to-one correspondence of S into $\mathcal{P}(S)$:

$$f : S \longrightarrow \mathcal{P}(S)$$

$$\begin{array}{ccc} \underline{S} & & \underline{\mathcal{P}(S)} \\ x & \longrightarrow & f(x) \text{ (a subset of } S) \end{array}$$

$x \in S \qquad f(x) \subseteq S$, or $f(x) \in \mathcal{P}(S)$.

(3) We're going to prove that there's no such f, by contradiction.

Consider $A = \{x \in S \mid x \notin f(x)\}$

($f(x)$ is called the image of x under f.)

(4) Claim: A is not the image of any $x \in S$ under f. [That would be a

contradiction, because we claimed that f was 1-1 onto $\mathcal{P}(S)$.]

Proof (by contradiction):

Suppose that A is the image of $x \in S$:

$$f(x) = A.$$

Is $x \in A$?

a. Suppose it is: $x \in A = f(x)$.

Since x is an element of A,

$x \notin f(x)$. But that's a contradiction!

b. Suppose it isn't:

$$x \notin A = f(x).$$

But therefore, since $x \notin f(x)$, by definition it's an element of A! But that's a contradiction!

Since x must either be an element of A or not, & since it can't be either,

x does not exist!

(5) There is no x that maps to A.

$\therefore$ A is not covered by f.

$\therefore$ f was not 1-1.

So there is no 1-1 mapping from S to $\mathcal{P}(S)$. $\boxed{S \text{ is smaller than } \mathcal{P}(S)!}$

(6) The Power Set has cardinality greater than the set itself: $\text{Card}(\mathcal{P}(S)) > \text{Card}(S)$.

(7) $\text{Card}(S) < \text{Card}(\mathcal{P}(S)) < \text{Card}(\mathcal{P}(\mathcal{P}(S))) < \ldots$

To ∞ + beyond!

Maybe A doesn't exist! But no....

A exists. Some element $x \in S$ must map to φ under f. $x \to \varphi$.

And $x \notin \varphi$. $\therefore$ $x \in A$.