Day 24, MAT115

- Announcements:
  - This session will be captured on Zoom, if I remember to turn it on, and record it.
    Our page of zooms and the play-by-plays.
  - On Friday I am striking for the climate, in association with Fridays for Future.

    I will be up on the 4th floor atrium, near the central elevators, during class time (and, in fact, for much of the day -- 9am-3pm), and I will have information about why the Climate, and the damage humans are doing to it, is worth a school strike.

    Come and visit me during class time, and I will give you a chance to win a dollar off of me at Fibonacci Nim (and I hope to lose a lot of dollars).
  - Your Egyptian multiplication and division homework is due today. (submit on Canvas)
  - You have a new Reading assignment (regarding Fibonacci numbers):

    Please read Chapter 10: *Working Your Quads*; also this short description of the Fibonacci numbers

    I'll soon be posting a new IMath assignment, which will check to see if you're able to win at Fibonacci Nim.

    But there will also be some throwback questions, to make sure that you're getting ready for our next exam.
- Last time:
  - We did one last review Egyptian division problem, which illustrated that you may have to combine repeated fractions, and use the unit fraction table multiple times to get your final answer (which includes only whole numbers and unit fractions, without repeating any unit fractions).
  - Then Great Fraudini was able to hold onto his hard-earned cash (**twice**!), even though his opponents in the game "Fraudini Nim" held his fate in their hands. How did he do it?
- Today's Question of the Day:

  What is the third and last great factorization, and how can it make me a million dollars?

  I. Destination Fibonacci...
     - Recall that the Great Fraudini used the binary factorization to read your minds. However astonishing (and even though that might win you a **little** Easter candy), the next factorization will provide you with a game that you can win almost every time, if you're playing with the uninitiated.

       This is like having a candy factory!:)
     - Before we win a million dollars (always promising, never delivering...), however, let's begin with an old Greek problem about "common divisors". This is a problem considered by the great Greek mathematician Euclid.

       What's the greatest common divisor of 33 and 21?

       What's the answer, and how do you know?

       We can see that our first great factorization comes into play here (or could); but we're going to see that we can think about the problem **geometrically** (and we're preparing to move this course into the geometric realm, and away from the number realm).
     - To get at Euclid's problem, we're going to work with this sheet, which features rectangles on both sides. Let's start with the side that has two rectangles on it.

       A. The objective is two-fold:
          1. to "geometrize" the problem -- turn it into something that Greeks three thousand years ago could see; and
          2. to turn a big problem into successively smaller problems, which end in the solution. I.e., turn it into a problem where we can "do it again", over and over, and finally land on the solution.

       B. We start with the geometry: you see before you a large rectangle of dimensions 33x21. If you must tile the whole rectangle with the largest squares possible, how big are the squares?

       C. What does the size of the squares have to do with the greatest common divisor (GCD)?

       D. From the given rectangle, can you make a smaller rectangle that is also tiled by exactly the same sized squares?

       E. The answer is yes, and the trick is to "cast out" a ______________ of size ______________.

       F. Now we have another rectangle with the same GCD. What are its dimensions?

       G. What do we do now? **Do it again**....
     - Now we'll do the same thing with another example:

       What's the greatest common divisor of 33 and 6?

       Why does this one go so much faster?
     - What would make this process go **as slowly as possible**, for **as long as possible**? (This is the key question, which leads us to the Fibonacci numbers!) Two things:
       1. You can only cast out one square at a time, and
       2. The largest tile has size 1x1.
     - Now turn over your page:

       What's the greatest common divisor of 34 and 21?

       Carry out the procedure. The numbers that appear are called the Fibonacci numbers, after the great Leonardo de Pisa -- aka "Fibonacci".
     - Let's write down the succession of square sizes: these are the famous "Fibonacci numbers".

       Question: What's the next bigger Fibonacci number after 34?
     - What's the pattern?

  II. Back to "Fraudini Nim", and the secrets of success.
     - Rules:
       - There are two players, Player 1 and Player 2.
       - An arbitrary number of counters is thrown down (e.g. toothpicks, pennies).
       - Objective: **The player who picks up the final counter wins.**
       - Player 1 will go first. On the first play, Player 1 must pick up at least one counter, but may not pick up all the counters (otherwise this would be a silly game!).
       - Player 2 must then pick up at least one counter, and not more than twice the number of counters picked up on Player 1's turn.
       - The players then alternate turns, subject to these same conditions: pick up at least one, no more than twice what the preceding player took.
       - The player who legally picks up the final counter on their turn wins.
     - Let's remind ourselves of what we know so far, by looking at some of the simplest cases:

Number of counters	1	2	3	4	5	6	7	8
Winner (1 or 2? -- if they play right!)	X	2	2	1	2	1	1	2
First player pick	X			1		1	2	

     - What's the winning strategy in Fraudini Nim? It is based off of **the third great factorization**, the "Fibonacci factorization". It turns out that

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       Every natural number is either
       1. Fibonacci, or
       2. it can be written as a sum of **non-consecutive** Fibonacci numbers in a unique way.

       Examples?
       1. 34 is Fibonacci
       2. 38 = 34 + 3 + 1
       3. 98 = 89 + 8 + 1
     - Now, for the secrets of Fibonacci Nim. You need to learn this strategy, and you can't lose.

       Suppose there are n counters on the table to start.
       1. First or Second? To be "in the driver's seat", you should
          - arrange to go **second** if n is Fibonacci;
          - go first otherwise.
       2. Your move? If you're in the driver's seat, then you'll be able to do this:
          - First of all, if you can legally take all the counters, do so! Then you win. Otherwise,
          - Write n as the unique sum of non-consecutive Fibonacci numbers; call the smallest Fibonacci m.
          - **Take m counters.**

          You'll be guaranteed a victory.

          You'll know that you're **not** in the driver's seat if you can't follow this strategy. The reason why? Remember that you can only take up to **twice** what your opponent took. Although you might write the number of counters remaining as a sum of Fibonacci numbers, you will find that you are not allowed to take even the smallest of the Fibonaccis -- it will be **more than twice** what your opponent picked up, so that would be illegal.

       Let's look at some examples:
       - 29 counters, and you're first. You're in the driver's seat, since 29 isn't Fibonacci.

         29 = 21 + 8   6 = 5 + 1

         so you take 8.

         I'm up, and looking at 21. It's already Fibonacci. I can't take the smallest in the "sum" -- 21. So I'm out of luck. I can't follow "the strategy". Whatever I take, you'll be able to beat me.

         To slow things down, I take 1.

         It's now 20 to you. 20 = 13 + 5 + 2. So you take 2 (that's legal). And so on, and you're going to beat me. Argh!
       - 34 counters, and you're second. You're in the driver's seat, since 34 is Fibonacci.

         Suppose I go first, and take 4.

         Then it's 30 to you. 30 = 21 + 8 + 1, so you take 1. Now it's 29 to me.

         You might think "29 to Long, and he can pick first -- he's going to win!" But that's wrong. I'd like to take 8, as before, but I can't -- because 8 is more than twice what you took.

       Now here's **another key point**: **when you're not in the driver's seat, take 1** -- the smallest legal move. That's because you want to slow down the game, and let your opponent make a mistake and put you back in the driver's seat.
     - So basically, the whole things comes down to this. If you can choose whether to go first or second, do it and you're guaranteed a victory.
       1. Go **first for non-Fibonacci, second for Fibonacci.**
       2. Of course, **take all of the counters is that's a legal move** (this is what Fraudini could have done when playing Becky -- but he took one, to keep the game going).
       3. Whatever the number of counters on the table, write it as a sum of non-consecutive Fibonacci numbers;
          1. **take the smallest Fibonacci, if you can**; you're now in the driver's seat, and should win.
          2. **if you can't** take the smallest (because it's more than twice your opponent's move), **take 1** -- it will slow the game down, and hopefully your opponent will make a mistake.

- **Links**:
  - Unit Fraction Table
  - How Egyptians wrote fractions.
  - Problem 81: converting between different measures with fractions
  - Egyptian fractions (Wikipedia)
  - An amazing source for Egyptian Fraction info
  - The new Mayan glyphs from Xultun, Guatemala: lots of spectacular calendar calculations, but nothing whatsoever about 2012 AD
  - "Ancient Maya Astronomical Tables from Xultun, Guatemala". William A. Saturno, David Stuart, Anthony F. Aveni, Franco Rossi
  - Part of the wall from which our Mayan tablet is taken
