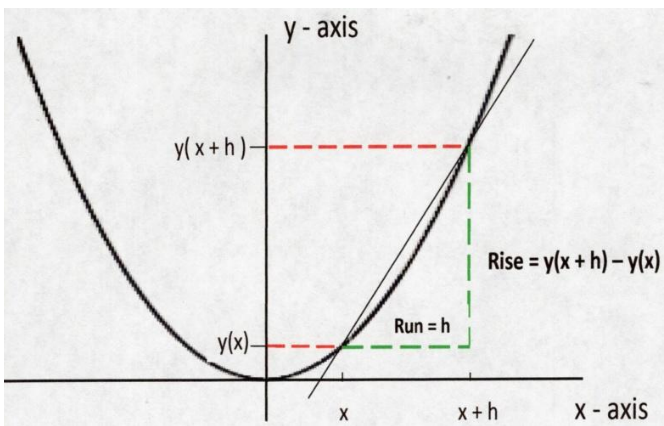
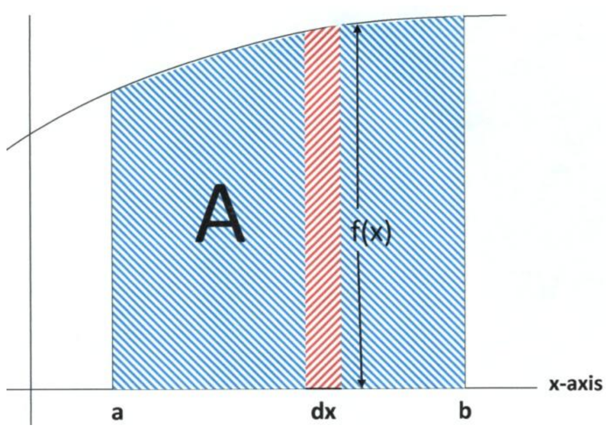
The Little Book of Calculus

Don Kelly Jacob Kelly Benjamin Kelly





"Just do the exercises diligently, then you will know what you have understood, and have not." Arnold Sommerfeld

INTRODUCTION

The Little Book of Calculus started in 2019 as a 4-page self-help booklet. It has grown but has very modest goals: develop the two basic features of calculus, the derivative and the integral. Five units make up the presentation.

Unit 1 presents review material (Straight Lines, Slope, Exponents, Logarithms and Functions). Unit 2 develops the limit concept and Unit 3 presents the derivative – the key concept in calculus. Unit 4 introduces the integral, the second major feature of calculus. Unit 5 is a salute to Leonhard Euler, his number (**e**) and his magical formula.

There are numerous examples. Working through the examples will help you understand the material. Simply reading them won't work. The exercises are designed to reinforce the examples. Find the time to work on each exercise. You will be rewarded. An Appendix provides answers and solutions for most exercises.

Calculus is a lot like arithmetic, algebra and sports. The more you practice the better you get. Understanding new ideas takes practice. The exercises are your practice fields. There is plenty of blank space on the backs of the pages facing those with exercises – makes a great place to work.

Every great city has its own language. In Paris it's French. In Rome it's Italian. In Oxford, Ohio it's English. Math is not a city but it has its own language. We call it Math Speak and you will find bits and pieces of it sprinkled on these pages.

Thanks to Jim Williams, Dr. Ellen Buerk, Dr. John Howard and Dr. Joe Kennedy for their insights and help.

Feedback Welcome. We'd love to hear from you. don@donckelly.com

Oxford, OH October 28, 2021 Calculus Rules!

This guide is online at www.donckelly.com/MATH/MATH_TOPICS.html

CONTENTS

UNIT 1 REVIEW

Straight Lines

Let's start with a question. If you make a point on a page like this one, how many straight lines can you draw though it? Now add a second point. How many straight lines can you draw that pass through both points? How many points are needed to uniquely specify a straight line? (Answers below*)

We use a Cartesian coordinate system to visualize lines. Rene Descartes (1596 – 1650) invented the coordinate system which bears his name. It uses of two perpendicular lines, called the x-axis and the y-axis. Figure 1 shows a Cartesian system with axes labeled by numbers. In Figure 1 the red point is at x = 1, y = 3, abbreviated (1,3). The blue point is at x = -4, y = -3 (-4, -3). The point (0,0) is called the **origin**.

* Infinity of lines, One line, Two points.

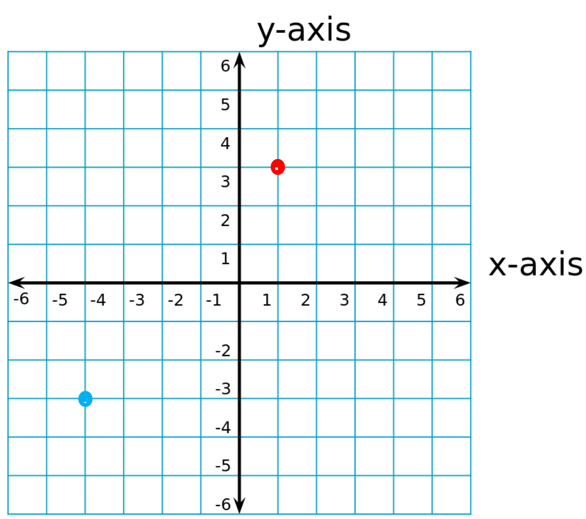


Figure 1. Cartesian coordinates

Exercise 1. Locate (5, 2) in Figure 1.

Slope

An important property of a line is its **slope**, a quantity that measures how steep the line is. Here's how the slope is defined: Pick two points [A (-2,-4) & B (2,4) in Figure 2] and measure the **rise** (the **change in y** between A&B) and the **run** (the **change in x** between A&B). In Figure 2, the rise is 8 (y changes from -4 to +4), the run is 4 (x changes from -2 to +2). The slope is defined as the ratio of rise to run. The slope of the line in Figure 2 equals 2.

$$\text{slope} = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x} = \frac{8}{4} = 2$$

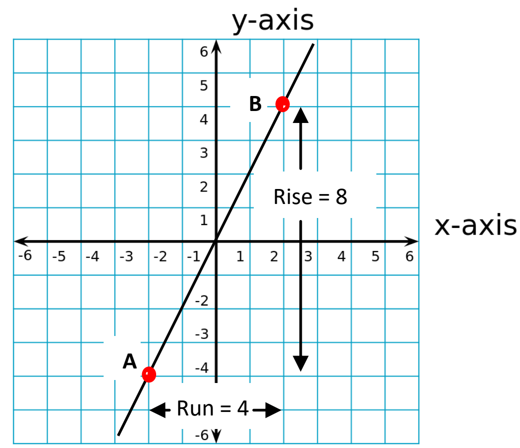


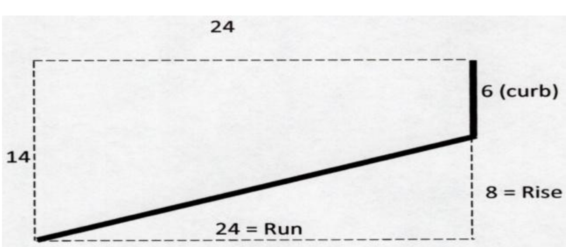
Figure 2. Slope = rise/run = 8/4 = 2

Different choices of points may give different values for the rise and run. The value of the slope is the same for all choices. The slope of a straight line is a special property. Hold that thought.

Exercise 2. Pick a different pair of points on the line in Figure 2 and work out the slope.

Curb Cuts, Slope and the Law

Patterson's Café is located in Oxford, Ohio. It is a popular dining destination situated on a corner. The logical spot for a handicap accessible curb cut would be close to the doors leading into Patterson's. The actual curb cut is 80 feet away. This irritates you if you are hungry and in a wheel chair. The logical position, close to the doors, would be ILLEGAL. The law (Americans with Disability Act (ADA) 4.8.2 Jan 26, 1992) requires that the **slope** of the surface at the curb cut be no greater than $\frac{1}{12}$. This means that the rise of the surface can be no more than one inch over a run of one foot. At Patterson's a point 24 inches from the curb is 14 inches above the pavement. The curb is 6 inches high. This gives a rise of 8 inches for a run of 24 inches (figure below) and a slope of $\frac{8}{24} = \frac{4}{12}$. The slope at Patterson's is greater than $\frac{1}{12}$ so the curb cut had to be moved.



When you moved from arithmetic to algebra you learned to use letters to represent numbers. In algebra **we use an equation to describe a straight line**. The equation uses letters to represent the coordinate values (x,y) for points on the line.

Values of x and y at points on a straight line are related by the **Straight Line Equation**

$y = \mathbf{m} \cdot x + \mathbf{b}$ **Straight Line Equation**

x and y are the variables that locate points on the line. The quantities **m** and **b** are constants. Different values of **m** and **b** give different lines. Figure 3 shows three (x,y) points that lie on the line

$y = 2x + 1$ $\mathbf{m} = 2$ $\mathbf{b} = 1$

You choose a value for x and use $y = 2x + 1$ to find the value of y

$x = -1$ $y = 2(-1) + 1 = -1$ $(-1, -1)$

$x = 0$ $y = 2(0) + 1 = 1$ $(0, 1)$

$x = 1$ $y = 2(1) + 1 = 3$ $(1, 3)$

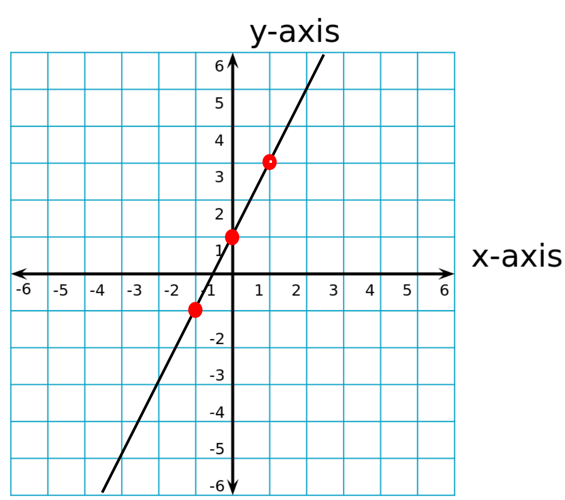


Figure 3

Sure enough, all three points lie on a straight line. The slope of the line is 2.

Exercise 3. In Figure 3, plot three points for the line

$$y = \frac{3}{2}x - 3$$

Do your points lie on the same line? What is the slope of your line?

The slope of the line y = **m**·x + **b** equals **m**. To prove this we pick two points on the line, (x1,y1) and (x2,y2). The rise is y2 – y1. The run is x2 – x1. Using the straight line equation we get

$$y2 - y1 = \mathbf{m}\cdot x2 + \cancel{b} - (\mathbf{m}\cdot x1 + \cancel{b}) = \mathbf{m}\cdot(x2 - x1)$$

Dividing both sides by (x2 – x1) shows that the slope equals **m**

$$\text{slope} = \frac{y2-y1}{x2-x1} = \mathbf{m}$$

If you set x = 0 in the equation y = **m**·x + **b** you get y = **b**. This shows that **b** is the value of y where the line crosses (intercepts) the y-axis. **b** is called the "y-intercept".

To summarize: The equation

y = **m**·x + **b** **Straight Line**

describes a straight line with slope **m** and y-intercept **b**.

Look at the three lines in Figure 4. The green line D passes through (0,1) and (5,4). The rise is 3 (y changes from 1 to 4), the run is 5 (x changes from 0 to 5), so the slope is $\frac{3}{5}$. Its y-intercept is 1. Its equation is

$$y = \frac{3}{5}x + 1 \qquad (\mathbf{m} = \frac{3}{5}, \mathbf{b} = 1)$$

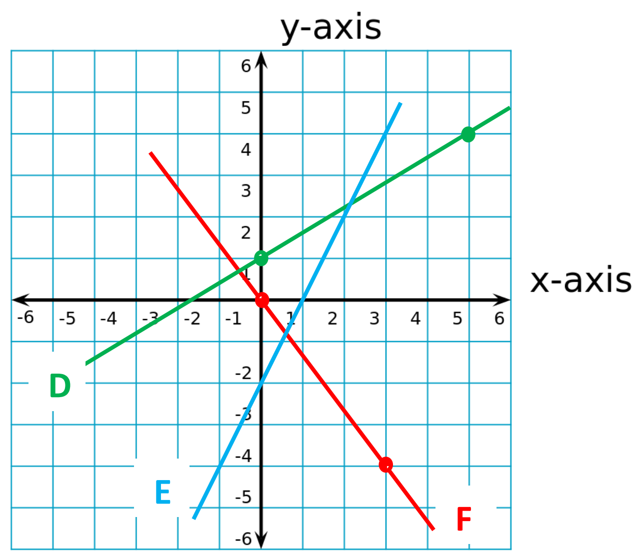


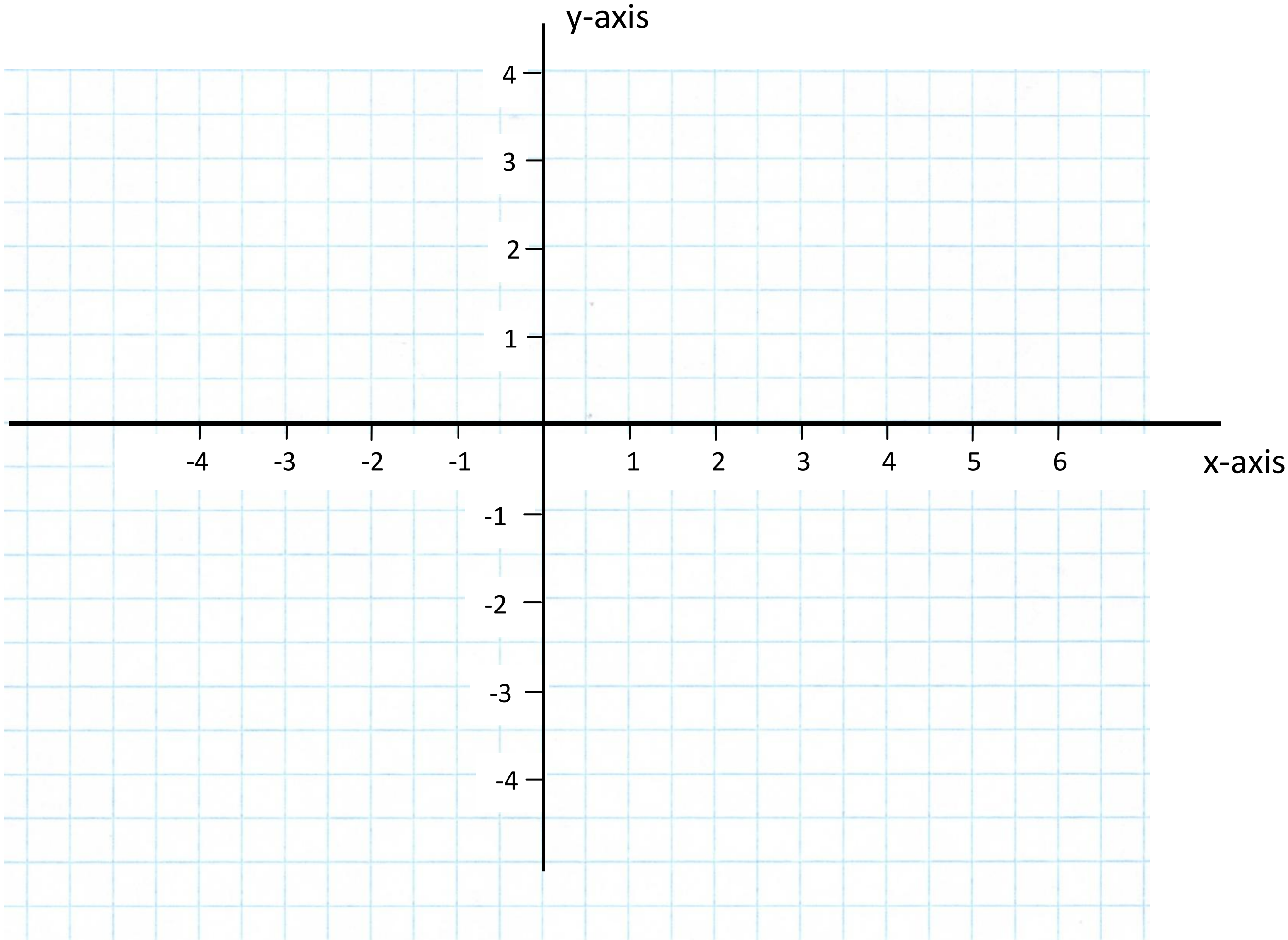
Figure 4

The red line F passes through the origin (0,0) so its y-intercept is zero (**b** = 0). It also passes through (3,-4) so its rise is -4 and its run is 3. Its slope is $\mathbf{m} = -\frac{4}{3}$. Its equation is $y = -\frac{4}{3}x$.

Exercise 4. The red line $y = -\frac{4}{3}x$ in Figure 4 passes through a point where $x = \frac{3}{2}$. What value should y have at that point? Does the figure confirm that value?

Exercise 5. What is the slope (**m**) of the blue line in Figure 4? Its y-intercept (**b**)? Write the equation for the line. Does (2,2) lie on the line?

Straight Line Review Exercises



Exercise 6. Using the coordinate system above, draw a straight line through the points (-1,-3) and (5,3). Use the two points to determine its slope. Slope = _______

Determine the slope again, using the points (2,0) and (4,2). Do you get the same value?

Exercise 7. Using the coordinate system above, plot the line y = 3. What is its slope? If a line is horizontal its slope = ______ .

Exercise 8. Using the coordinate system above, draw the line whose equation is $y = \frac{1}{2}x + 2$. Confirm that it passes through the points (0, 2) and (2,3). What is its slope? Its y-intercept?

Exercise 9. Using the coordinate system above, draw straight lines whose y-intercept equals -2 and whose slope equals A) 1/4 B) 0 C) -1/2. Write the equation for each line.

EXPONENTS

An exponent tells you how many times 1 is multiplied by a number

$1 \cdot 10 \cdot 10 = 10^2$, $\quad 1 \cdot x \cdot x \cdot x = x^3 \quad 1 \cdot x \cdot x \cdot x \cdot x = x^4$

This definition makes $\mathbf{x^0 = 1}$.

Multiplying by 1 does not change anything so we can drop the "1" factor.

Exponents make math more efficient.

$x^0 = 1 \quad x = x^1 \quad x \cdot x = x^2$ Efficiency not evident

$x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$ Efficiency becoming evident

$\underbrace{x \cdot x \cdot x \cdot x \cdot x \cdots\cdots x \cdot x \cdot x}_{\text{107 times}} = x^{107}$ OK, OK, I got it

Negative exponents count how many times 1 is **divided** by a number

$x^{-1} = \frac{1}{x} \qquad x^{-2} = \frac{1}{x^2} \qquad x^{-3} = \frac{1}{x^3}$ etc.

The **product rule**: The exponent of a product is the sum of the exponents of the factors.

$$\mathbf{x^a \cdot x^b = x^{a+b}}$$

In words: If you multiply 1 by x **a** times and then multiply the result **b** times you have multiplied a total of **a + b** times.

Example. $2^3 \cdot 2^2 = 2^{3+2} = 2^5$

$2^3 \cdot 2^2 = (2 \cdot 2 \cdot 2)(2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$

The **power rule** is a consequence of the product rule: $\mathbf{(x^a)^b = x^{ab}}$

$(x^3)^2 = (x \cdot x \cdot x)(x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$

$(2^3)^2 = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2) = 2^6$

Question? If we use $5 \cdot 5 = 25$ as an example what name do we use to describe 5?

A) root canal B) Route 66 C) square root

Using exponents that are fractions is a neat way of writing roots.

Example. The square root of x ($\sqrt{x}$) is defined by $\sqrt{x} \cdot \sqrt{x} = x$. The product rule

$$(x^{1/2})(x^{1/2}) = x^1 = x$$

shows that $x^{1/2}$ is the square root of x.

$$x^{1/2} = \sqrt{x}$$

Exercise 10. The cube root is defined by

$$\sqrt[3]{x} \cdot \sqrt[3]{x} \cdot \sqrt[3]{x} = x$$

Which quantity equals the cube root of x?

A) $x^{3/1}$ B) $x^{1/3}$ C) $x/3$ D) $x - 1/3$ E) $x + 1/3$

Using fraction exponents makes it easy to evaluate products involving roots.

Example.

$(\sqrt{2}\,\sqrt[3]{2})^6 = (2^{1/2}\, 2^{1/3})^6 = (2^{1/2+1/3})^6 = (2^{5/6})^6 = 2^5 = 32$

$(\sqrt[5]{2}\,\sqrt[10]{2})^{10} = (2^{1/5}\, 2^{1/10})^{10} = (2^{3/10})^{10} = 2^3 = 8$

Exercise 11. Evaluate the following:

$A)\ (\sqrt{2}\,)^2$ B) $((2^{1/3})^2)^3$ C) $(2^{3/4})^{2/3}$ D) $2^{-3}\cdot 2^3\cdot 2^5$

E) $((2^2)^3)^2$ F) $((2^3)^2)^2\cdot 2^{-12}$ G) $(2^2\cdot 2^3\cdot 2^{-5})^{14}$

LOGARITHMS

Logarithms are deluxe exponents. Logarithms are exponents but they don't have to be integers or fractions. Logarithms let us represent numbers in terms of a **base**. When we write 10^3 we are using base 10 and 3 is the logarithm. When we write 2^4 we are using base 2 and 4 is the logarithm. The format is

$$\text{Number} = \text{Base}^{\text{Logarithm}}$$

If we call the number A and its logarithm logA the relation says

$$A = \text{Base}^{\log A}$$

Example. The number 1024 can be written as 2^{10} or $10^{3.0103}$ or $\mathbf{e}^{6.93147}$. The base **e** refers to Euler's number e ≈ 2.71828. Base e logarithms are called "natural" logarithms and are designated by "ln".

$$\ln 1024 = 6.93147 \quad \Rightarrow \quad 1024 = e^{6.93147}$$

Math Speak: Euler is pronounced "Oiler".

Calculators have a "log" key for base 10. The "ln" key is for base e.

Exercise 12. Complete the table.

Number	Base 2	ln	log
8	3		
64			
4096			3.612

Example. Comparing

$1 = \text{Base}^{\text{Log } 1}$ with $1 = x^0$

shows that the logarithm of one equals zero for all bases.

log 1 = 0 (All Bases)

Logarithms and exponents obey the same rules. The **product rule** for logarithms: The logarithm for the product of two numbers is the sum of the logarithms of the individual numbers.

$\log(A\cdot B) = \log A + \log B$ **Product Rule**

Example. $\log(4\cdot 10^2) = \log 4 + 2 = 2.604$

$\ln(2e^3) = \ln 2 + 3 = 3.693$

$\log 8000 = \log 8 + \log 1000 = 0.903 + 3 = 3.903$

The product rule $\log(AB) = \log A + \log B$ can be used to prove

$\log(A^N) = N \log A$ **Power Rule**

$\log\left(\frac{A}{B}\right) = \log A - \log B$ **Quotient Rule**

Example. $\log A^2 = 2\log A$

$\log\sqrt{5} = \log 5^{1/2} = ½\log 5 = ½(1.876) = 0.938$

If two quantities are equal their logarithms are equal

$$K = M \Rightarrow \log K = \log M$$

This lets you solve problems by "taking logs", i.e. equating the logarithms of both sides.

Example. Solve for x by "taking logs".

$$4\cdot 10^x = 10^{2x} \Rightarrow \log(4\cdot 10^x) = \log(10^{2x})$$

The product rule gives $\log(4\cdot 10^x) = \log 4 + x$ and $\log(10^{2x}) = 2x$ so equating logs gives

$$\log 4 + x = 2x \quad \Rightarrow \quad x = \log 4 = 0.604$$

If the logarithms of two quantities are equal then the quantities are equal

$$\log K = \log M \quad \Rightarrow \quad K = M$$

Example. Solve for x.

$$\log(x+1) - \log(x-1) = \log 2$$

The Quotient Rule shows that the left side equals $\log(\frac{x+1}{x-1})$ so we have

$$\log(\frac{x+1}{x-1}) = \log 2 \quad \Rightarrow \quad (\frac{x+1}{x-1}) = 2 \quad \Rightarrow \quad x = 3$$

Exercise 13. Solve for x. $\ln(x+3) - \ln(x-1) = \ln 3$

Example. Solve for x. $2\log(x-1) = \log 25$

The power rule $\log(A^N) = N\log A$ shows that $2\log(x-1) = \log(x-1)^2$ so we have

$$\log(x-1)^2 = \log 25 \Rightarrow (x-1)^2 = 25$$

Taking the square root of both sides gives

$$x - 1 = \pm 5 \quad \Rightarrow \quad x = 6, -4$$

Only x = 6 is valid. The solution x = -4 does not satisfy $2\log(x-1) = \log 25$. If x = -4 log(x-1) equals log(-5). Alas, negative numbers don't have logarithms. Don't take our word for it, ask your calculator for the value of log(-5). Setting $2\log(x-1) = \log(x-1)^2$ introduced the invalid solution.

Example. Given $\log K = 3.1 \quad \log M = 2.4$

Find A) log(K/M) B) $\log\sqrt{KM}$ C) KM

A) $\log(K/M) = \log K - \log M = 3.1 - 2.4 = 0.7$

B) $\log\sqrt{KM} = ½\log KM = ½(\log K + \log M) = 2.75$

C) $KM = 10^{\log KM} = 10^{\log K + \log M} = 10^{5.5}$

Exercise 14. Given $\log K = 4 \quad \log M = 2$ find

A) KM B) $\log(K\sqrt{M})$ C) $\log(K^2M)$

Question? Are logarithms and exponents the same thing? Answer: Basically, YES! It's OK to use either word.

Relation	**Example**
$\text{Number} = \text{Base}^{\text{Logarithm}}$	$8 = 2^3 \quad 1000 = 10^3$
$\log(AB) = \log A + \log B$	$\log 6 = \log 3 + \log 2$
$\log A^N = N\cdot\log A$	$\log 5^3 = 3\log 5$
$\log(\frac{A}{B}) = \log A - \log B$	$\log(\frac{6}{10}) = \log 6 - \log 10$
$\log 1 = 0$	

Logarithm Relations (All Bases)

FUNCTIONS

The word "function" has several meanings. A math function acts like a machine that receives an input and produces an output. For our purposes, **a function is a rule that relates two numbers**. We use an equation to express the rule. For example, the function

$$f(x) = x^2$$

describes the rule "Square the argument*". It relates values of x and x^2 . The value of x is the input, the value of x^2 is the output.

***Math Speak**: "f(x)" is pronounced "eff of ex". The symbol x in f(x) is called the **argument** of the function.

The argument can be a number. For $f(x) = x^2$

$f(2) = 2^2 = 4$

$f(5) = 5^2 = 25$

Evaluating $f(x) = x^2$ is easy when the argument is a number. **There will be many instances where the argument is not a number**. For example, what is f(x+3)? The argument is x + 3. The rule is "Square the argument", so

$f(x+3) = (x + 3)^2$

Changing the argument from a number to an expression like x + 3 may present a conceptual hurdle. Hurdles come one at a time. Let's get over the first one.

Example. If f(x) = 2x + 7, evaluate f(5) and $f(x^2)$. The rule is this: "Multiply the **argument** by 2 and then add 7."

$f(5) = 2{\cdot}5 + 7 = 17$ (That was easy)

Here comes that first hurdle. You must understand the rule " f(x) = 2x + 7 ".

$f(x^2) = 2x^2 + 7$ (Did you clear the hurdle?)

The argument is x^2, so you multiply x^2 by 2 and then add 7.

Exercise 15. For each value of x, fill in the missing values of the functions f(x), g(x), y(x) and z(x).

x	f(x) = 2x – 7	g(x) = 1/x	$y(x) = x^2$	z(x) = x + 3
-1	-9		1	
0		div by zero		3
2		½		
5	3			8
10			100	

Exercise 16. If $y(x) = \sqrt{x}$, which option completes y(x + h) = ?

A) $\frac{1}{x}$ B) $\frac{1}{x+h}$ C) $\sqrt{x+h}$ D) $\sqrt{x} + h$

Exercise 17. If y(x) = sin x, which option completes y(x+h) = ?

A) sin(x+h) B) cos(x+h) C) x+h D) x + sin h

Exercise 18. Write an equation for a function f(x) described by "Multiply the cube of the argument by 4 and add 0.55". Evaluate it when the argument equals A) 3 B) z^2.

Exercise 19. The area of a circle is related to its radius. Write an equation for the function A(r) that expresses the relation "Area equals π (pi) times the square of the radius (r)".

Example. The function f(x) is defined by

f(x) = 2x + 1 => multiply the argument by 2 and add 1

Evaluate **f(**f(3)**)**. The quantity f(3) is the argument. We first must evaluate f(3):

f(x) = 2x + 1 => f(3) = 2·3 + 1 = 7

Now we can evaluate

$$f(f(3)) = f(7) = 2\cdot 7 + 1 = 15$$

A different approach gives the same result

f(f(3)) = 2·f(3) + 1 = 2·7 + 1 = 15

Exercise 20. Let $y(x) = 1 + x + x^2$

What is the value of y(y(1)) ?

A) 1 B) 3 C) 7 D) 13

Exercise 21. For the function

$$Z(x) = \frac{5x+2}{3x+8}$$

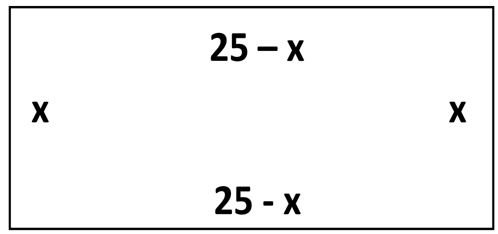
what value of x makes Z(x) = 1?

Hint: Set Z(x) = 1 and solve for x.

Example. The perimeter of a rectangle is 50 cm. Show that the area of the rectangle, A(x), can be expressed by $A(x) = 25x - x^2$ where x is the length of one side.

If x is the length of one side an adjacent side has a length 25 – x. The area of a rectangle equals its length times its height, so

$$A(x) = (25 - x)x = 25x - x^2$$



There is one value of x that makes A(x) a maximum. Can you guess or figure it out?

We will revisit this maximum question again when we develop the derivative. Figuring out the maximum value or minimum value of functions is one of many applications where calculus rules!

Exercise 22. Celebrating Beta $\boldsymbol{\beta}$

Let $\beta(x) = x + 7$

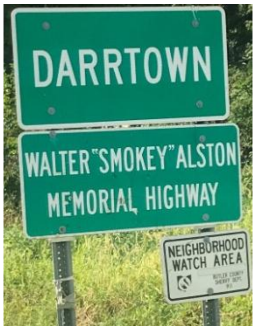
A) Evaluate $\beta(5)$

B) Evaluate $\beta(3) + \beta(2)$

C) Solve for x: $\beta(x) + \beta(x+3) = 9$

UNIT 2 LIMITS

LIMITS (The Road to Darrtown)



Darrtown, Ohio is located 8 miles from Oxford, Ohio. A cricket sets out from Oxford and walks half way to Darrtown. She stops for the night and the next morning decides to walk half the remaining distance before stopping for the night. She repeats this over and over, walking half the remaining distance each day. The cricket can see the city limits sign for Darrtown. She gets closer and closer. **She knows where the limit is even though she never reaches it.** Math limits are similar. You can figure out what they are without actually reaching them.

To illustrate a typical limit problem let's look at the ratio R(h)

$$R(h) = \frac{(3+h)^2 - 9}{h}$$

We want to figure out the limit of R(h) as h approaches zero. We can't do this by setting h equal to zero because that gives zero divided by zero

$$R(0) = \frac{(3+0)^2 - 9}{0} = \frac{9-9}{0} = \frac{0}{0}$$

We can prove that the limit value is 6 by expanding the binomial in the numerator.

$$(3 + h)^2 = 3^2 + 6h + h^2 = 9 + 6h + h^2$$

This converts R(h) into

$$R(h) = \frac{(3+h)^2 - 9}{h} = \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h}$$

The nines cancel and **we factor h, which is then canceled by the h in the denominator**

$$R(h) = \frac{\cancel{h}(6+h)}{\cancel{h}} = 6 + h$$

Canceling the factor h gets rid of the divide by zero issue.

The equation R(h) = 6 + h shows us that R(h) approaches 6 as h approaches zero. The symbolism used for the limit process is

$$\lim_{h\to 0} R(h) = \lim_{h\to 0} (6 + h) = 6$$

Math Speak: The limit of 6 + h as h **approaches** zero equals 6.

Like the cricket who can see the city limits sign without reaching it, we can see the limit of 6 + h without setting h = 0.

Example. Evaluate $\lim_{h\to 0} \frac{6}{(6+h)}$

There is no divide by zero issue. The denominator approaches 6 as h approaches zero and the ratio approaches $\frac{6}{6} = 1$.

Exercise 23. Evaluate the limits

A) $\lim_{h\to 0} \frac{4}{(1+2h)}$ B) $\lim_{h\to 0} \frac{(8x+h^2)}{(4+3h)}$

C) $\lim_{h\to 0} \frac{(4+h)^2 - 16}{h(4+3h)}$ D) $\lim_{N\to\infty} (1 + \frac{1}{N})$

UNIT 3 DIFFERENTIAL CALCULUS

Slope and Derivative

We spent considerable time on the slope concept in Unit 1. Why is the **slope** so important? The short answer is that the

slope leads to the **derivative**. The derivative is the **key to the kingdom of calculus – it's our way in!**

A short answer is not enough. We start by showing how a **tangent line is defined in terms of a limit**. This will lead us to the derivative.

Secants and Tangents

Figure 5 shows an oval and several straight lines. All of the lines except the red one pass through the oval at two points – they are called **secants**. The red line touches the oval at one point – it is called a **tangent** line.

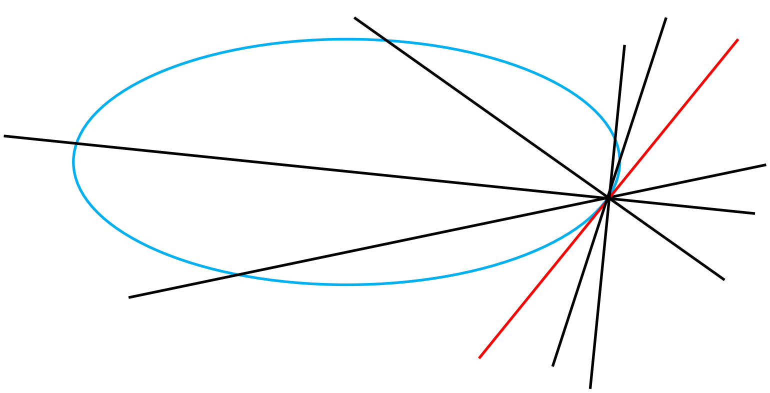


Figure 5 A tangent line touches at one point

The tangent line can be viewed as the limit of a sequence of secants whose two points come closer and closer together and meet at the tangent point.

Figuring out the slope of a tangent line was one of the problems that led to the invention of calculus by Gottfried Wilhelm Leibniz (1646 – 1716) and Isaac Newton (1643-1727). It resulted in a new type of function – called a **derivative**.

What is a Derivative?

The word "derivative" means "derived from" Our world is full of derivatives. Words have derivatives, chemical compounds have derivatives, the stock market is knee-deep in derivatives. Derivatives are everywhere. We want to get a handle on mathematical derivatives.

A derivative is a function that has a parent. You start with a function – perhaps one that describes an oval or other shape. From this parent you derive the function that describes its tangent line – the child. The tangent line function is a derivative of the function you started with. When it came time to choose a name for the child, "derivative" won out over "tangent line function".

Small Research Project: Stock **options** are one type of financial derivative. How do options work? In what sense are they derivatives?

We begin our derivative quest by seeing how a tangent line is defined in terms of a limit.

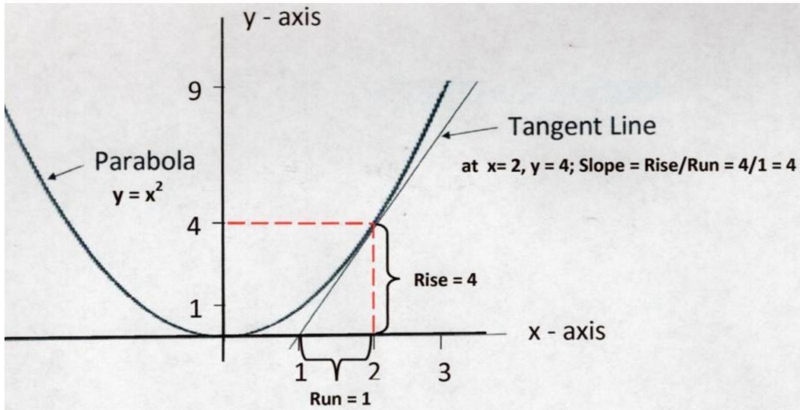


Figure 6. The slope of the tangent line at x = 2, y = 4 equals Rise/Run = 4/1 = 4

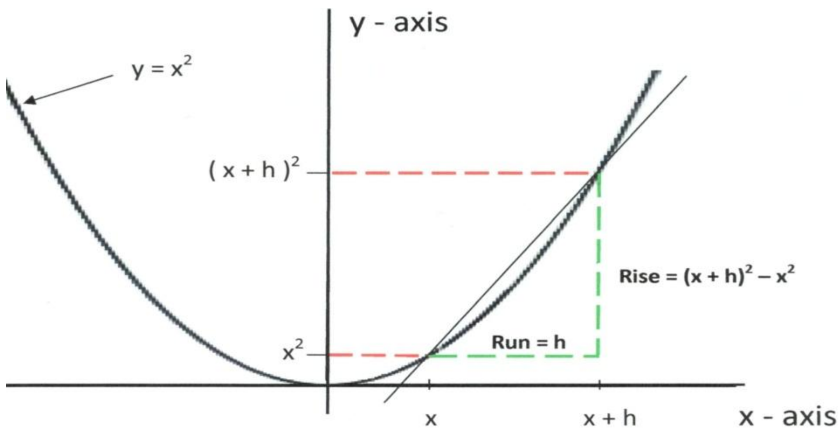


Figure 7. The graph is a parabola. The x-axis and y-axis have labels, not numbers.

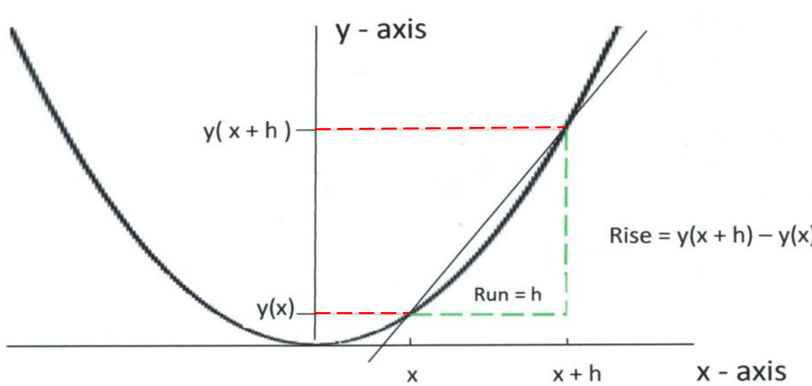


Figure 8. The labels on the y-axis refer to a function named y(x).

Its form is not specified, it could be any function.

Figure 6 shows a parabola ($y = x^2$) and a tangent line. The slope of the tangent line equals 4 at x = 2. The slope changes as you move to different points along the parabola.

Figure 7 looks similar to Figure 6. We are dealing with the same parabola ($y = x^2$). There is an important difference. Instead of numbers along the x-axis and y-axis we have labels. This will let us work out an equation for the slope of the tangent line at any point along the parabola. Sneak Peak! The equation is: **slope of tangent line = 2x**
As x changes the slope changes – the slope of the tangent line is not a constant.

On the x-axis in Figure 7 one point is labeled x so the y-value of the corresponding point is $y = x^2$. The second point on the x-axis is labeled x + h so the y-value of the point is $y = (x + h)^2$. We draw a secant line through the two points. Its slope is

$$\text{slope} = \frac{\text{Rise}}{\text{Run}} = \frac{(x+h)^2 - x^2}{\text{h}} = \frac{\cancel{x^2} + 2\text{xh} + h^2 - \cancel{x^2}}{\text{h}}$$

The x^2 terms cancel and we can factor h so the slope becomes

$$\text{slope} = \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2\text{x} + \text{h}$$

Next: **Imagine a sequence of such secant lines** with smaller and smaller values of h. As h is made smaller in Figure 7 the two points where the lines intersect the parabola come closer together. The tangent line is **DEFINED** as the limit of the sequence of secant lines as h approaches zero.

$$\text{Slope of tangent line} = \underset{h\to 0}{Lim} (2\text{x} + \text{h}) = 2\text{x}$$

This equation lets us determine the slope at any point on the parabola. It confirms the result shown in Figure 6, slope = 4 at x = 2.

Next, we **generalize**. We consider an arbitrary function, y(x), and work out an equation for the slope of its tangent line. This leads to a new function called the **derivative.**

Figure 8 looks very similar to Figure 7. But notice there are new labels along the y-axis. We don't specify the form of **y(x)** – it could be **any function**.

The slope of the secant line intersecting the graph at x and x + h is

$$\text{slope} = \frac{\text{Rise}}{\text{Run}} = \frac{\text{y(x+h)} - \text{y(x)}}{\text{h}}$$

As before, we imagine a sequence of such lines with smaller and smaller values of h. The limit of the sequence of secant lines becomes the tangent line for the function y(x). The limit of the slope **defines** a function symbolized as $\boldsymbol{(\frac{dy}{dx})}$.

$$\left(\frac{dy}{dx}\right) = \underset{h\to 0}{Lim} \frac{\text{y(x+h)} - \text{y(x)}}{\text{h}}$$ **Derivative Definition**

The symbol $(\frac{dy}{dx})$ is called the **derivative of y(x)**. The derivative determines the slope of the tangent line for the function y(x).

y(x) => original function

$(\frac{dy}{dx})$ => the derivative of y(x)

$\boldsymbol{(\frac{dy}{dx})}$ is a symbol for a new function, the derivative

Math Speak: "Differentiation" is the name for the procedure that converts a function into its derivative.

$\left(\frac{dy}{dx}\right)$ Why the Weird Symbolism?

The symbol $\left(\frac{dy}{dx}\right)$ looks like "dee times wy divided by dee times ex". It is not! The symbol $\left(\frac{dy}{dx}\right)$ encourages us to think of the derivative as a slope – a rise dy over a run dx. That's a good way to **visualize** a derivative. But … $\left(\frac{dy}{dx}\right)$ is not division – it's a **symbol** that reminds us the derivative is a slope. The parentheses are part of the symbolism. One last point: The quantities "dy" and "dx" are symbols. "dy" is not "d times y". We'll return to dx and dy later in this unit. For now, remember, $\left(\frac{dy}{dx}\right)$ is a symbol, not division.

Every function has its own derivative. They go hand in hand. Let's work out (derive!) the derivatives for a few functions.

We just worked out the equation for the derivative of the function $y(x) = x^2$

$$\left(\frac{dy}{dx}\right) = 2x \qquad \text{for} \quad y(x) = x^2$$

Let's start with the definition

$$\left(\frac{dy}{dx}\right) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} \qquad \textbf{Derivative Definition}$$

and work out the derivative for the straight line function, $y(x) = \mathbf{m}\cdot x + \mathbf{b}$. The derivative determines the slope so we expect to get $\left(\frac{dy}{dx}\right) = \mathbf{m}$ because **m** is the slope of the straight line. For $y(x) = \mathbf{m}\cdot x + \mathbf{b}$

$$\left(\frac{dy}{dx}\right) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h\to 0} \frac{\mathbf{m}(x+h) + b - (\mathbf{m}x + b)}{h}$$

$$= \lim_{h\to 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h\to 0} \frac{m\cancel{h}}{\cancel{h}} = \mathbf{m}$$

As expected, $\left(\frac{dy}{dx}\right) = \mathbf{m}$ for $y(x) = \mathbf{m}\cdot x + \mathbf{b}$

Next, let's work out the derivative for the function $y(x) = \frac{1}{x}$. We start from the definition $\lim_{h\to 0} \frac{y(x+h) - y(x)}{h}$ with $y(x) = \frac{1}{x}$.

In arithmetic you learned to handle fractions by setting up a common denominator. For example, $\frac{1}{2} - \frac{1}{3} = \frac{1\cdot 3}{2\cdot 3} - \frac{1\cdot 2}{3\cdot 2} = \frac{3-2}{6} = \frac{1}{6}$. We have fractions whose denominators are algebraic expressions, not numbers. The common denominator here is $x\cdot(x + h)$

$$y(x+h) - y(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x\cdot 1}{x\cdot(x+h)} - \frac{1\cdot(x+h)}{x\cdot(x+h)}$$

$$= \frac{x-(x+h)}{x\cdot(x+h)} = \frac{-h}{x\cdot(x+h)}$$

Divide by h and take the limit to get the derivative

$$\left(\frac{dy}{dx}\right) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}\cdot x(x+h)}$$

$$\left(\frac{dy}{dx}\right) = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

$$\left(\frac{dy}{dx}\right) = -\frac{1}{x^2} \quad \text{for} \quad y(x) = \frac{1}{x}$$

Exercise 24. For the function $y(x) = x^3$, start from the derivative definition and show that $\left(\frac{dy}{dx}\right) = 3x^2$. You will need the binomial expansion for the cube of (x + h),

$$(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

DERIVATIVE RULES. Here are three rules that apply to all derivatives:

Rule 1) If C is a constant,

$$\left(\frac{d\, C\cdot f(x)}{dx}\right) = C\cdot\left(\frac{df}{dx}\right)$$

For example, if $f(x) = x^2$ and C = 10 then

$$\left(\frac{d\, 10\cdot x^2}{dx}\right) = 10\cdot\left(\frac{d\, x^2}{dx}\right) = 10\cdot 2x = 20x$$

To prove this relation we start from the definition of the derivative,

$$\left(\frac{dC\cdot f}{dx}\right) = \lim_{h\to 0} \frac{C\cdot f(x+h) - C\cdot f(x)}{h} = \lim_{h\to 0} \frac{C[f(x+h) - f(x)]}{h}$$

The constant C can be factored because it does not depend on h

$$\left(\frac{dC\cdot f}{dx}\right) = C\cdot \lim_{h\to 0} \frac{[f(x+h) - f(x)]}{h} = C\cdot\left(\frac{df}{dx}\right) \quad \text{Done!}$$

Exercise 25. Use Rule 1 and Table 1 to work out the derivative.

A) $g(x) = 10x^3$ B) $f(x) = 5e^{4x}$

Rule 2) The derivative of a sum is the sum of the individual derivatives,

$$\frac{d\{f(x) + g(x)\}}{dx} = \left(\frac{df}{dx}\right) + \left(\frac{dg}{dx}\right)$$

Example. The function $y(x) = x^2 - 8x$ is the sum of x^2 and $-8x$. Its derivative is $2x - 8$, the sum of the derivatives of x^2 and $-8x$.

$$\frac{d\,(x^2 - 8x)}{dx} = \frac{d\,(x^2)}{dx} + \frac{d(-8x)}{dx} = 2x - 8$$

Derivative of sum = sum of derivatives

Exercise 26. Use Rule 2 and Table 1 to work out the derivative of

A) $5x^2 + 2x + 7$ B) $x^{13} + x^7 + x^{-7} + 7$

Rule 3) The Product Rule

$$\frac{d\{f(x)\cdot g(x)\}}{dx} = f(x)\left(\frac{dg}{dx}\right) + g(x)\left(\frac{df}{dx}\right)$$

Example. The function $y(x) = x^2\cdot e^{3x}$ is the product of $f(x) = x^2$ and $g(x) = e^{3x}$. Plug

$$f(x) = x^2 \quad \left(\frac{df}{dx}\right) = 2x \qquad g(x) = e^{3x} \quad \left(\frac{dg}{dx}\right) = 3e^{3x}$$

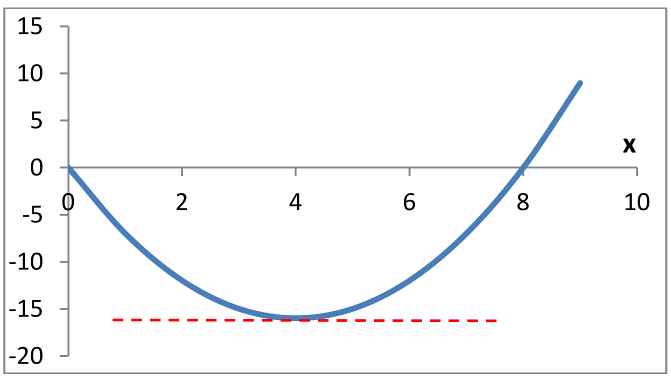
into the product rule. This gives

$$\frac{d\{x^2 e^{3x}\}}{dx} = x^2\cdot 3e^{3x} + e^{3x}\cdot 2x = (3x^2 + 2x)e^{3x}$$

Exercise 27. Use the Product Rule to find the derivative of x^4. Hint: Treat x^4 as the product of x^2 and x^2. Table 1 may be helpful.

Using Derivatives: How To Find A Minimum

The figure shows a graph of the function $y(x) = x^2 - 8x$. We want to use the derivative to locate the value of x where $x^2 - 8x$ reaches its minimum.



There are two pieces to the puzzle: i) The tangent line - - - - - - - - is horizontal at the minimum so **its slope is zero**. ii) The slope equals the derivative, $\left(\frac{dy}{dx}\right)$. We combine these two ideas to get a prescription for finding the value of x for which y(x) is a minimum:

Set $\left(\frac{dy}{dx}\right)$ equal to zero and solve for x

For $y(x) = x^2 - 8x$ $\quad \left(\frac{dy}{dx}\right) = 2x - 8 = 0$

The minimum, where the slope is zero, is at $x = 4$.

The same method lets you find positions where a function has a **maximum.** The tangent line at a maximum is horizontal so its slope (the derivative) is zero.

Exercise 28. Find the value of x where the function $y(x) = 6x - x^2$ has a maximum. Set $\left(\frac{dy}{dx}\right) = 0$ and solve for x. Use an electronic calculator to plot $6x - x^2$. Does the plot confirm your value of x?

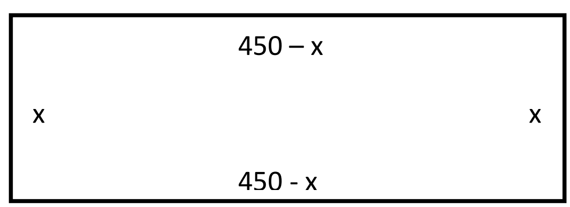
Exercise 29. For the function

$$y(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x$$

locate values of x where the slope is zero. Hint: $\left(\frac{dy}{dx}\right) = x^2 - x - 6$ can be factored. Use your calculator to confirm your values.

The takeaway: **Setting $\left(\frac{dy}{dx}\right)$ equal to zero and solving for x lets you locate maxima and minima.**

Example A farmer has 900 feet of fencing to enclose a rectangular field. What dimensions will enclose the largest area?
Solution: Let x denote the length of one side. An adjacent side has a length of $450 - x$. The area is $A(x) = (450 - x)x = 450x - x^2$



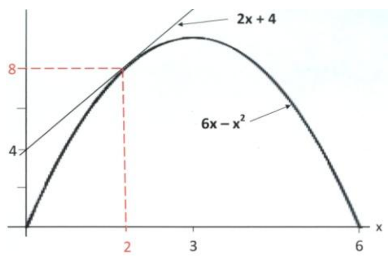
Set $\left(\frac{dA}{dx}\right) = 0$ and solve for x.

$$\left(\frac{dA}{dx}\right) = 450 - 2x = 0 \quad \Rightarrow \quad x = 225$$

This value makes the figure a square.

Exercise 30. The farmer in the example above has another field adjacent to a pond. He wants to use his 900 feet of fencing to enclose a rectangular field that uses the pond as an open side. Find the dimensions which maximize the area of the field.

Example. We want to show that $y = 2x + 4$ is the equation of the straight line that is tangent to the parabola $y = 6x - x^2$ at $x = 2$.



The straight line equation is

$y = \mathbf{m} \cdot x + \mathbf{b}$ $\quad$ **m** => slope, **b** => y-intercept

We must figure out the values of **m** and **b**.

The slope of the tangent line (**m**) equals the value of the derivative at the tangent point. The derivative of $6x - x^2$ is $6 - 2x$, so **m** equals the value of $6 - 2x$ at $x = 2$

$\mathbf{m} = 6 - 2x$ (at $x = 2$) $= 6 - 2 \cdot 2 = 2$

The slope is $\mathbf{m} = 2$. We now have $y = 2x + \mathbf{b}$ for the tangent line.

To figure out **b** we use the fact that y has the same value for the parabola and the line at the tangent point. At $x = 2$ on the parabola, y has the value 8,

$y = 6x - x^2$ (at $x = 2$) $= 6 \cdot 2 - 2^2 = 8$

The value of y on the line also equals 8 at $x = 2$.

$y = 8 = 2x + \mathbf{b}$ (at $x = 2$) $= 2 \cdot 2 + \mathbf{b}$ => $\mathbf{b} = 4$

The equation of the tangent line is

$$y = 2x + 4$$

Use an electronic calculator to plot $2x + 4$ and $6x - x^2$. Look good?

Exercise 31. Determine the equation for the straight line which is tangent to the parabola $y = x^2$ at the point $x = 3$. Hint: Start with

$y = \mathbf{m} \cdot x + \mathbf{b}$

As in the preceding example, you will need two equations to determine **m** and **b**. A sketch may help. Use a calculator to graph your tangent line and the parabola.

Table 1 presents the derivatives of selected functions. We've derived some of the results in Table 1. The others will be derived as we move ahead. Table 1 is repeated on the back cover.

y(x)	$(\frac{dy}{dx})$	y(x)	$(\frac{dy}{dx})$
$m \cdot x + b$	m	e^x	e^x
x^2	$2x$	e^{kx}	$k \cdot e^{kx}$
x^3	$3x^2$	$e^{f(x)}$	$e^{f(x)}(\frac{df}{dx})$
x^N	$N x^{N-1}$	$\sin x$	$\cos x$
$\ln x$	$\frac{1}{x}$	$\cos x$	$- \sin x$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\tan x$	$[\sec x]^2$

Table 1. Derivatives of selected functions

Example. Evaluate the derivative of the function

$f(x) = x^{3} + x^{-3}$

For what **positive value of x** does f(x) have a minimum? What is the minimum value?

From Table 1 we note that the derivative of x^N is Nx^{N-1}. Use this with N = 3 and N = -3

$$\left(\frac{df}{dx}\right) = \frac{d(x^3+x^{-3})}{dx} = 3x^2 - 3x^{-4}$$

To locate the minimum you set the derivative equal to zero and solve for x.

$3x^2 - 3x^{-4} = 0 \;\; => \;\; x^6 = 1 \;\; => \;\; x = 1$

The minimum value is $f(1) = 1^3 + 1^{-3} = 2$.

Exercise 32. Use Table 1 to evaluate the derivative of the function

$g(x) = x^{2} + 16x^{-2}$

For what positive value of x is g(x) a minimum? Determine the minimum value.

We have used $\left(\frac{dy}{dx}\right) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}$ to work out derivatives of several functions. Let's look at a powerful **indirect** method for figuring out derivatives.

The Chain Rule

Suppose we wanted to differentiate the function $(x^2 + 5)^3$. There is no entry in Table 1 for the function $(x^2 + 5)^3$. The **chain rule** provides a way to evaluate derivatives of more complex functions in terms of derivatives of elementary functions like those in Table 1.

The chain rule introduces the idea of a **function whose argument is another function**.

For example, we can set $z(x) = (x^2 + 5)$ and write $(x^2 + 5)^3 = z^3 = f(z)$. **The argument of the function f(z) is the function z(x)**.

$x => z(x) => f(z(x))$

z(x) is an intermediate function that connects x and f(z(x)).

The chain rule states that $\left(\frac{df(z(x))}{dx}\right)$ can be expressed as the product of two derivatives:

$$\left(\frac{df(z)}{dx}\right) = \left(\frac{df(z)}{dz}\right)\cdot\left(\frac{dz}{dx}\right) \qquad \textbf{Chain Rule}$$

$\uparrow$ x $\qquad \uparrow$ z

The chain rule lets us evaluate the left side by chaining together the two derivatives on the right side. To illustrate the chain rule we use it to evaluate the derivative of $(x^2 + 5)^3$.

For the intermediate function z(x) we choose

$z(x) = x^2 + 5$ We need its derivative

$$\left(\frac{dz}{dx}\right) = \frac{d(x^2+5)}{dx} = 2x$$

In terms of z the function $(x^2 + 5)^3$ equals z^3

$f(z) = z^3$

Its derivative with respect to z is

$\left(\frac{d\,z^3}{dz}\right) = 3z^2$ [Table 1, Exercise 24]

Plugging into the chain rule equation we get

$$\frac{d\,(x^2+5)^3}{dx} = \left(\frac{d\,z^3}{dz}\right)\left(\frac{dz}{dx}\right) = 3z^2 \cdot 2x = 6x(x^2+5)^2$$

We replaced z^2 by $(x^2 + 5)^2$ in order to express the final result in terms of x.

Using the chain rule we were able to express the derivative of $(x^2 + 5)^3$ in terms of the derivatives of the more elementary functions z^3 and $x^2 + 5$. A proof of the chain rule is given in Unit 5.

Exercise 33. Use the chain rule to show that the derivative of $(x^2 + 6x)^4$ equals $4(2x+6)(x^2 + 6x)^3$. Use $z = (x^2 + 6x)$.

Example. Reciprocal Derivatives The chain rule lets us prove a useful property of derivatives. Start with the chain rule $\left(\frac{df}{dx}\right) = \left(\frac{df}{dy}\right)\left(\frac{dy}{dx}\right)$ and set f(x) = x.
Then $\left(\frac{df}{dx}\right) = \left(\frac{dx}{dx}\right) = 1$ and the chain rule becomes $1 = \left(\frac{dx}{dy}\right)\left(\frac{dy}{dx}\right)$. This proves

$$\boldsymbol{\left(\frac{dx}{dy}\right) = \frac{1}{\left(\frac{dy}{dx}\right)}}$$

$\left(\frac{dy}{dx}\right)$ and $\left(\frac{dx}{dy}\right)$ are reciprocals.

Exercise 34. Use the chain rule to differentiate A) $f(x) = (3x^2 - 5x)^{15}$

B) $y(x) = \sin(x^4)$ C) $g(x) = (x^4 + 2x^3 + 5)^3$

Cyber Security and Inverse Functions

The buzzwords "cyber security" are a reminder that secrecy is fragile. Messages are encrypted before transmission and decrypted at the receiving end, restoring them to their original form. The tandem of encryption-decryption is an example of a function and its **inverse**. We use the notation $f^{-1}(x)$ to denote the inverse of f(x).

Math Speak: f^{-1} is pronounced "eff inverse".

The inverse function "undoes" whatever f(x) does. **Applying f^{-1} to f(x) restores x**.

If x is the message and f(x) is the encrypted message then f^{-1} describes the decryption function that restores x. Symbolically,

x ->	f(x)	->->->	$f^{-1}(f(x))$	-> x
message	encryption	transmission	decryption	message

The basic equation relating f and f^{-1} is

$$\mathbf{f^{-1}(f(x))} = x$$

For example, the inverse of $f(x) = x^2$ is $f^{-1}(x) = \sqrt{x}$. We can use words or equations to illustrate the basic equation $f^{-1}(f(x)) = x$.

Words: the square root of (the square of x) = x

Equation: $f^{-1}(f(x)) = \sqrt{f(x)} = \sqrt{x^2} = x$

Reversing the order also works

$$\mathbf{f(f^{-1}(x))} = x$$

For $f(x) = x^2$ we have

Words: the square of (the square root of x) = x

Equation: $\mathbf{f(f^{-1}(x))} = (f^{-1}(x))^2 = (\sqrt{x}\,)^2 = x$

Exercise 35. The letters in the English alphabet are assigned numerical codes:

A = 1, B = 2,..... Z = 26

Encryption uses the "key" $f(x) = x^2$

$A = 1^2 = 1$, $B = 2^2 = 4$, ... $Z = 26^2 = 676$

Here is the encrypted message: 9 1 144 9 441 144 441 361 324 441 144 25 361

What is the message?

Logarithms and Exponential Growth

The connection between exponents and logarithms was developed in Unit 1.

10 times 10 became 10^2, x·x·x became x^3

John Napier invented logarithms in 1614 and the genie was out of the bottle. They became mathematical functions, not mere numbers. We often deal with "natural" logarithms and the associated exponential function. The base for natural logarithms is Euler's number

e = 2.71828…

Figure 9 shows graphs of the exponential functions 2^x, e^x and 3^x. Their dramatic upturns illustrate "exponential growth".

Exponential growth results when successive values increase by the same factor. The values form a geometric series. By contrast values of quantities which increase **linearly** form an arithmetic series. Figure 9 also shows the linear growth of the function y1 = x.

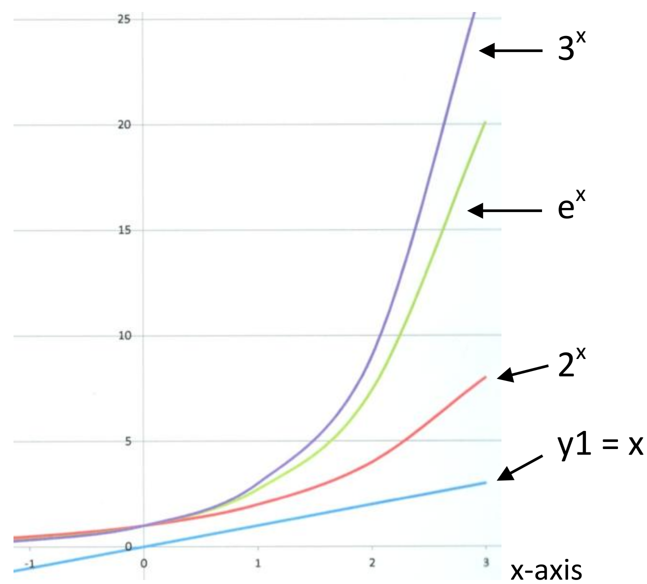


Figure 9. Exponential growth

Exercise 36. As the variable x increases, which of the following quantities show exponential growth? Linear growth?

A) 2x B) 2^x C) 2^{2x} D) 2x + 2 E) e^{4x}

The Rule of 72

Ted opens a savings account. He deposits $100 and is pleased to learn that his account will earn 4% interest **compounded annually**. Ted calculates the future value of his savings on a year by year schedule.

Initial Value ($) 100

Value after 1 year $100(1 + 0.04) = 104$

Value after 2 years $104(1 + 0.04) = 100(1 + 0.04)^2 = 108.16$

Value after 3 years $108.16(1 + 0.04) = 100(1 + 0.04)^3 = 112.99$

Value after 4 years $112.99(1 + 0.04) = 100(1 + 0.04)^4 = 116.99$

.....

Value after N years $100(1 + 0.04)^N$

The value at the end of a year is multiplied by (1 + 0.04) to get the value at the end of the next year. After N years Ted's savings have increased by a factor of $(1 + 0.04)^N$. The values show "exponential growth". The values form a geometrical series which increase annually by the factor (1 + 0.04).

At 4% Ted's investment increased by a factor of (1 + 0.04) each year. More generally if the interest rate is **r** (%) then the yearly increase factor would be $(1 + \frac{r}{100})$. After N years the initial value would be multiplied by $(1 + \frac{r}{100})^N$.

$$\text{Value at N years} = (\text{Initial Value})\cdot(1 + \frac{r}{100})^N$$

Ted wants to test the "Rule of 72" which states that the product of the interest rate (in %) and the number of years for an investment to **double in value** equals 72. At a rate of 8% the Rule of 72 says it would take 9 years to double in value (8x9 = 72).

(Interest Rate)(Years to double) = 72 **Rule of 72**

We want to show that the rule of 72 is an approximation that makes it easy to do the math in your head.

The mathematical basis for the rule of 72 starts with the factor $(1 + \frac{r}{100})^N$. When this factor equals 2 the investment has doubled.

$$(1 + \frac{r}{100})^N = 2$$

This relates r and N for doubling. If you pick a value for N you can use a calculator to find r. If you pick a value for r you can use a calculator to find N. Start with $(1 + \frac{r}{100})^N = 2$, take the natural log of both sides. This gives

$$N \ln(1 + \frac{r}{100}) = \ln 2 = 0.693$$

Example. Credit card debt compounds at a rate of 18%. How many years would it take for the debt to double? With r = 18 the value of N is 0.693/ln(1.18) = 4.19.

To avoid logs we use the approximation

$$\ln(1 + \frac{r}{100}) \approx \frac{r}{100}$$

This converts $N \ln(1 + \frac{r}{100}) = 0.693$ into

$$Nr \approx 69.3$$

(See Exercises 61 and 62)

This says the number of years to double (N) times the interest rate (r) equals 69.3. Over the traditional range of interest rates the value 72 works better than 69.3

$Nr \approx 72$ **Rule of 72**

Our example gave N = 4.19 for r = 18. The Rule of 72 gives N = 4. The rule of 72 is an approximation that lets you do exponential growth math in your head.

Exercise 37. How long does it takes to double your savings at 4% interest. Use the Rule of 72.

Exercise 38. Ted wins \$1024 in a small lottery. He spends half of his winnings on a new laptop. The next day he spends half of his remaining winnings on a watch. He continues, spending half of his remaining money each day. How many days does it take for his money to reach \$1?

Exponential Decay and Half Life

Ted's story is an example of exponential "decay". The amount of money he has after N days of spending can be expressed as $\frac{\$1024}{2^N}$ After 1 day he has half the original \$1024. After two days he has one-fourth, after three days one-eighth, etc. As N increases from 1 to 2 to 3 the denominator 2^N increases from 2 to 4 to 8, etc. The denominator increases exponentially, the remaining amount of money decreases exponentially.

Check your answer to the exercise above. 2^{10} equals 1024 so it takes 10 days for Ted's winnings to shrink to \$1.

Compound interest produces exponential growth. Ted's spending results in an exponential "decay" – the remaining amount is cut in half repeatedly.

Ted's spending leads to the concept of a **half life**. The **half life of a quantity is the time for its value to be halved**. The half life of Ted's winnings is 1 day.

Half Life => Time for value to be halved

Exercise 39. A Starbucks regular has two cups of coffee with her 6AM breakfast. The half life of caffeine in her body is 4 hours. What fraction of the caffeine remains when she sits down for supper at 6PM?

The equation that describes the exponential decay of Ted's lottery money is

$$\text{Money Remaining} = \frac{\$1024}{2^N}$$

Let's generalize: Let Q(t) denote the value of a quantity that decreases exponentially with the time t. We use Q(0) to denote its starting value.

Let T denote the half life. The ratio (t/T) is the number halvings in time t.

N = (t/T) => number of halvings in time t

The generalization of the equation describing Ted's money is

$$Q(t) = \frac{Q(0)}{2^{(t/T)}} = Q(0)2^{-t/T}$$

Setting $2 = e^{0.693}$ (e = 2.718..) converts Q(t) into the standard form of an exponential "decay"

$$Q(t) = Q(0)\, e^{-0.693(t/T)} \quad \textbf{Exponential Decay}$$

Radioactive Dating

Willard Libby (1902-1980) was an American chemist who developed a method that uses the decay of radioactive elements to determine the ages of ancient materials.

Libby used the decay of a radioactive isotope of carbon (carbon-14). Most carbon found in nature is in the form of carbon-12, a stable isotope. A tiny fraction is carbon-14. Living systems, especially plants, maintain a constant ratio of the two isotopes. When the plant dies the ratio of carbon-14 to carbon-12 decreases. The carbon-12 is stable, the carbon-14 decays, transforming into a different element. The decay is observed by detecting energetic electrons which are part of the decay products. Importantly, the half life of carbon-14 decay is known: It is approximately 5730 years.

Libby knew the rate at which carbon-14 decays in living matter and he measured the decay rate for ancient material. If his sample showed a decay rate that was half that for living matter he could conclude that the age of the sample was one half life (5730 years).

Libby confirmed his calibration by using ancient artifacts whose ages were known from historical records. These included wood from Egyptian tombs, material from the Dead Sea Scrolls, redwood trees from California and carbonized bread recovered from the ruins of Pompeii (Mt Vesuvius). Their known ages ranged from 2000 years to 5000 years.

Using isotopes of other elements scientists have determined ages ranging from a few years (wine) to billions of years (meteorites).

We'll work out the equation Libby used for radioactive dating after we study the derivative of the exponential function.

Exponential Functions and Derivatives

The exponential function is unique: The derivative of e^x equals e^x,

$$\left(\frac{\mathrm{d}\,e^{\mathrm{x}}}{\mathrm{dx}}\right) = e^{\mathrm{x}}$$

This relation is derived in Unit 5. There is nothing special about x, we could use any letter, for example, $\left(\frac{\mathrm{d}\,e^{\mathrm{f}}}{\mathrm{df}}\right) = e^{\mathrm{f}}$.

We can use the **chain rule** to prove a more general relation

$$\left(\frac{\mathbf{d}\,\boldsymbol{e}^{\mathbf{f(x)}}}{\mathbf{dx}}\right) = \boldsymbol{e}^{\mathbf{f(x)}}\left(\frac{\boldsymbol{df}}{\boldsymbol{dx}}\right)$$

f(x) can be any function. The proof is brief!

$$\underset{\substack{\uparrow\\ \mathrm{x}}}{\left(\frac{\mathrm{d}\,e^{\mathrm{f(x)}}}{\mathrm{dx}}\right)} = \underset{\substack{\uparrow\\ \mathrm{f}}}{\left(\frac{\mathrm{d}\,e^{\mathrm{f(x)}}}{\mathrm{df}}\right)}\left(\frac{df}{dx}\right) = e^{\mathrm{f(x)}}\left(\frac{df}{dx}\right)$$

This says: **The derivative of an exponential equals the exponential times the derivative of the exponent**.

An important example is the derivative of e^{kx} where k is a constant. The derivative of

f(x) = kx   is $\left(\frac{df}{dx}\right)$ = k   which gives

$$\left(\frac{\mathrm{d}\,e^{\mathrm{kx}}}{\mathrm{dx}}\right) = \mathrm{k}\cdot e^{\mathrm{kx}} \qquad \mathrm{k = constant}$$

There are many applications of calculus where the quantities of interest vary with **time**. The position of an object in motion changes with time. Its velocity is the rate at which its position changes. If y(t) denotes its position, its velocity is $\left(\frac{dy}{dt}\right)$. In the study of radioactive decay the number of nuclei changes with time. If Q(t) denotes the number of nuclei, the decay rate is $\left(\frac{dQ}{dt}\right)$. The **rate of change** is the time derivative, i.e. the derivative with respect to the time variable.

Q(t) => a function of time (t)

$\left(\frac{dQ}{dt}\right)$ => **rate** at which Q changes

For an exponential function like e^{kt} the rate of change also varies exponentially

$$\left(\frac{d\,e^{kt}}{dt}\right) = k\cdot e^{kt}$$

Mathematically $\left(\frac{d\,e^{kt}}{dt}\right)$ is the derivative of the function e^{kt}. Physically it is the **rate** at which the quantity e^{kt} changes.

We want to revisit radioactive dating. The decay of radioactive nuclei is described by an exponential

$Q(t) = Q(0)\, e^{-0.693(t/T)}$ **Exponential Decline**

The quantity which is measured is the rate of decay $\frac{dQ}{dt}$, not the number of nuclei. The decay rate is called the **activity** and is denoted by I(t)

$$I(t) = \frac{dQ}{dt}$$

Because the derivative of an exponential function is proportional to the function the activity follows the same exponential decay as the number of nuclei,

$I(t) = I(0)\, e^{-0.693(t/T)}$ Activity Decay

We want to solve this for t, the time the decay has progressed. Divide by I(0), take the natural log of both sides and solve for t

$$t = 1.44T\cdot\ln\left(I(0)/I(t)\right)$$

This equation let Willard Libby measure the ages of ancient materials. He measured I(t) the activity in the ancient sample and I(0), the activity of carbon-14 in living specimens. The half life for the decay of carbon-14 was known (T = 5730 years).

Example. The activity of a sample of wood from an Egyptian tomb is I(t) = 9.2 dpm/g (disintegrations per minute per gram). The value of I(0) is 13.6 dpm/g. The age of the specimen is

$$t = 1.44\cdot 5730\text{ yr}\cdot\ln(13.6/9.2) = 3220\text{ yr}$$

Exercise 40. The activity of the innermost layers (heartwood) of a giant redwood is 12.1 dpm/g. What is its age?

Exercise 41. As the variable x increases, which of the following quantities show exponential growth? Exponential decay? Linear growth?

A) $4x$ B) 4^{-x} C) 4^{x} D) e^{-4x} E) e^{4x}

Exercise H The half life of a chocolate bar on a store shelf is two days. The manager of the store does not want the supply to fall below one-eighth of the full level. How often does she have to restock the shelf?

Exercise T Technetium-99m (Tc-99m) is a radioactive isotope used in hospitals. It has a half life of 6 hours. What fraction of the Tc-99m remains in the patient 24 hours after being injected.

We move ahead to another aspect of exponentials and logarithms. If you ask your calculator for the value of ln 2 it responds with 0.693. This means that $2 = e^{0.693}$. We also could write this as $2 = e^{\ln 2}$. This form is a specific instance of the relation $x = e^{\ln x}$. This equation is an **identity***. It is true for all values of x. [*The functions e^x and ln x are inverses: $\ln(e^x) = e^{\ln x} = x$.]

Example We want to prove $(\frac{d \ln x}{dx}) = \frac{1}{x}$

Differentiate $x = e^{\ln x}$ and use the relation

$$\left(\frac{d\, e^{f(x)}}{dx}\right) = e^{f(x)}\left(\frac{df}{dx}\right)$$

$$\left(\frac{dx}{dx}\right) = 1 = \left(\frac{d\, e^{\ln x}}{dx}\right) = e^{\ln x}\left(\frac{d \ln x}{dx}\right) = x\left(\frac{d \ln x}{dx}\right)$$

This gives

$$\left(\frac{d \ln x}{dx}\right) = \frac{1}{x}$$

Exercise 42. Determine the value of x that minimizes the function

$g(x) = 3x^{-1} + \ln x$

What is the minimum value of g(x)?

Example. The 800-Pound Gorilla (Derivative)

The identity $x = e^{\ln x}$ and our derivative relation $\left(\frac{d\, e^{f(x)}}{dx}\right) = e^{f(x)}\left(\frac{df}{dx}\right)$ let us

prove an important derivative relation

$$\left(\frac{d\, x^N}{dx}\right) = Nx^{N-1} \qquad N = \text{constant}$$

Differentiate the identity $x^N = (e^{\ln x})^N = e^{N \ln x}$

and use the derivative relation

$$\left(\frac{d\, e^{N \ln x}}{dx}\right) = e^{N \ln x}\left(\frac{d\, N \ln x}{dx}\right)$$

This produces

$$\left(\frac{d\, x^N}{dx}\right) = e^{N \ln x}\left(\frac{d\, N \ln x}{dx}\right) \quad \text{Next, use}$$

$$e^{N \ln x} = x^N \text{ and } \left(\frac{d\, N \ln x}{dx}\right) = N\left(\frac{d \ln x}{dx}\right) = N\frac{1}{x}$$

The final result is $\left(\frac{d\, x^N}{dx}\right) = x^N N \frac{1}{x} = Nx^{N-1}$

$$\left(\frac{d\, x^N}{dx}\right) = Nx^{N-1}$$

This is the 800-pound gorilla of derivatives. Talk it out! The derivative of x^N equals N times x^{N-1}.

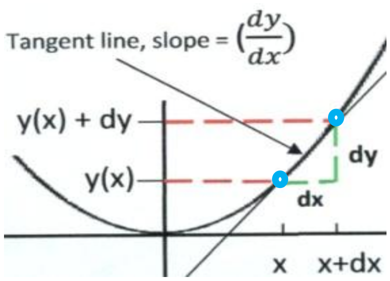
Exercise 43. Use $(\frac{d\,x^N}{dx}) = Nx^{N-1}$ to evaluate the derivatives of x^8, x^7, x^6 and x^5.

Exercise 44. If y = ln x then $x = e^y$. Use the reciprocal relation $(\frac{dy}{dx}) = \frac{1}{(\frac{dx}{dy})}$ to prove

$(\frac{d\ln x}{dx}) = \frac{1}{x}$

Differentials

The figure shows two points (x,y) and (x+dx, y+dy) and a tangent line. The quantities **dy** and **dx** are called



differentials (pronounced “dee wy” and “dee ex”). They are small changes in y and x. How small are differentials? We never have to answer that question numerically. As the figure suggests, the slope of the tangent line can be expressed as the ratio of the rise **dy** and the run **dx**. In fact, the ratio of the rise dy and the run dx is **defined** as the derivative – the slope of the tangent line. The equation expressing this definition is

$$\frac{dy}{dx} = (\frac{dy}{dx}) \qquad \text{Defines } \frac{dy}{dx}$$

dy divided by dx derivative

One more step: Multiply both sides by dx. This gives

$dy = (\frac{dy}{dx})\,dx$ **Differential Relation**

The dy on the left and the dy on the right are different. On the left dy is a differential, a small change in y. On the right dy is part of the derivative symbol. The same quantity, dy, has different meanings.

Exercise 44A The symbol dx appears twice on the right side. Explain how the two differ.

The original function y(x) relates **values** of x and y. The differential relation shows how the derivative relates **changes** in x and y.

y(x) relates **values** of x and y

$dy = (\frac{dy}{dx})\,dx$ relates **changes**, dx and dy

Example. For $y(x) = x^2$ $(\frac{dy}{dx}) = 2x$

The differential relation becomes

$$dy = (\frac{dy}{dx})\,dx = 2x\,dx$$

At x = 5, dy = 10 dx. The change in y is ten times the change in x. What is $(\frac{dy}{dx})$ at x = 10?

Exercise 45. A) Work out the equation for dy when y is the natural logarithm function y(x) = ln x.

B) Start with $u = (x - \ln x)^2$

Show that $du = 2(1 - \frac{1}{x})(x - \ln x)\,dx$

Hint: Use the chain rule: $(\frac{du}{dx}) = (\frac{du}{dz})(\frac{dz}{dx})$

Exercise 46. If $dy = 3x^2dx$, what is y(x)? Table 1 may help confirm your answer.

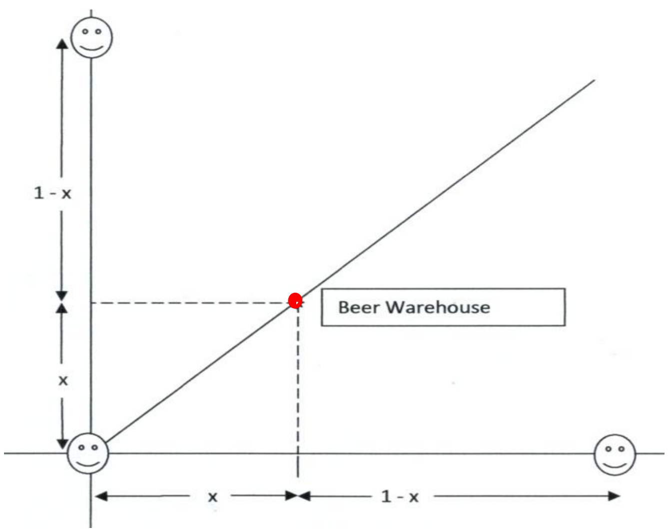
Exercise 46A: If dy = 5 dx what is the value of the derivative $(\frac{dy}{dx})$?

“Little Stones”

The Latin word “calculus” means “little stones”. Little stones can be put together to form a garden path. The differentials dx and dy can be put together to form the quantities x and y. The next unit, Integral Calculus, deals with putting differentials together.

Example. “Beer is proof God loves us” Ben Franklin

Three streets in a college town intersect as shown in the figure. Three bars are located in the town. One is at the intersection, a second is one mile north, the third is one mile east.



A distributor wants to locate a warehouse on the street which bisects the right triangle formed by the three bars. He wants to locate it so that the combined distances from the warehouse to the three bars is a minimum. This will minimize the cost of delivering beer.

The origin of our coordinate system is at the intersection. The positions of the three bars are (0,0), (0,1) and (1,0). The East-North coordinates of the warehouse are denoted as (x,x). We can express the distance from the warehouse to each bar in terms of x and add to get the combined distances.

The distance along the bisecting road from the bar at the intersection to the warehouse is $\sqrt{2}$ x. The other two bars sit at vertices of right triangles with sides x and 1 – x, making their combined distances from the warehouse $2\sqrt{2x^2 - 2x + 1}$

The combined distances for the three bars is

$$D(x) = \sqrt{2}\,x + 2\sqrt{2x^2 - 2x + 1}$$

To determine the minimum of D(x) you set its derivative equal to zero and solve for x. This leads to $x^2 - x + \frac{1}{6} = 0$. The solution is

$$x = \frac{1}{2} - \sqrt{\frac{1}{12}} = 0.211$$

Exercise 47. A lawn care service earns $800 per year for each of their 50 customers. They decide to expand. Each new customer reduces their net profit by $10 because of added equipment and transportation costs. Let x denote the number of new customers. Their total profit P(x) can be expressed as P(x) = (50 + x)(800 – 10x). What value of x maximizes their profit? What is the maximum profit?

Galileo and Lunar Astronauts

In 1589 Galileo performed his famous experiment. He dropped two objects of different weights from the Tower of Pisa. They fell side by side and reached the ground together. Galileo's experiment contradicted the popular idea of the time that heavier objects fall faster. Galileo also said "in the absence of air resistance a hammer and a feather fall equally fast."

In 1971 Apollo 15 Astronaut David Scott performed a demonstration on the Moon. He dropped a geologic hammer and a falcon feather at the same time. They fell "equally fast" and struck the lunar surface at the same moment. The hammer was 400 times heavier than the feather but gravity affected them equally.

You can view a video of the demonstration online.* Astronaut drops hammer and feather on the moon (freethoughtblogs.com) A second video shows an impressive drop on Earth, inside the world's largest vacuum chamber.

Position, Velocity and Acceleration

Derivatives help us work out the details of the motions on the moon and at the tower.

The motion is described in terms of three quantities: position, velocity and acceleration. We use y(t) to denote the position of the object at time t*. [* y = 0 and t = 0 mark the starting position and time.] The rate at which position changes is called the **velocity**. In terms of a derivative the velocity v(t) is the derivative of y(t).

v(t) = $(\frac{dy}{dt})$ => velocity = derivative of y(t)

Falling objects "speed up" as they fall – their velocity increases. We say they **accelerate**. Acceleration is defined as the rate at which velocity changes. In terms of a derivative the acceleration **a** is the derivative of v(t)

a = $(\frac{dv}{dt})$ => acceleration = derivative of v(t)

On the moon and on the earth **gravity produces a constant acceleration**. We use the letter **g** to denote the acceleration of gravity. The value of **g** for the earth is about six times the value for the moon

g = 9.8 m/s/s Earth

g = 1.6 m/s/s Moon

The acceleration is constant - as objects fall they speed up at a constant rate. The acceleration is the derivative of velocity so we can write

$(\frac{dv}{dt})$ = **g** = constant

Next, we want to figure out how v(t) depends on t. The derivative ($\frac{dv}{dt}$) is the slope of a graph of v versus t. The slope is constant so a graph of v versus t is a straight line. (See figure) The slope for the earth is 9.8 m/s/s. The slope for the moon is 1.6 m/s/s. The equation for v is

v(t) = **g**t **g** => slope (acceleration)

Next we want to figure out how y(t) depends on t. The velocity equals the derivative ($\frac{dy}{dt}$) so we can write

($\frac{dy}{dt}$) = **g**t

The derivative of t^2 equals 2t so y(t) must be proportional to t^2. In fact

y(t) = ½ $\mathbf{g}t^2$

Exercise 47A Show that y = ½ $\mathbf{g}t^2$ leads to ($\frac{dy}{dt}$) = **g**t.

The figure shows graphs of y = ½$\mathbf{g}t^2$ for the earth and moon.

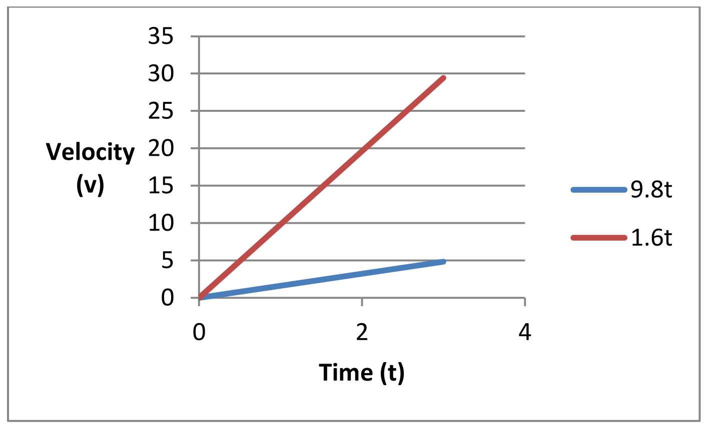
Example. We want to figure out two things. 1) How long did it take for the hammer and feather to fall and 2) what was their speed just before impact? To determine the time we use y = ½ $\mathbf{g}t^2$. For Scott's lunar drop the hammer and feather fell 1.6 m. The value of **g** on the moon is 1.6 m/s/s. Plug these values into the equation y = ½$\mathbf{g}t^2$ and solve for t

1.6 = (½)1.6 t^2 => t = 1.4 second

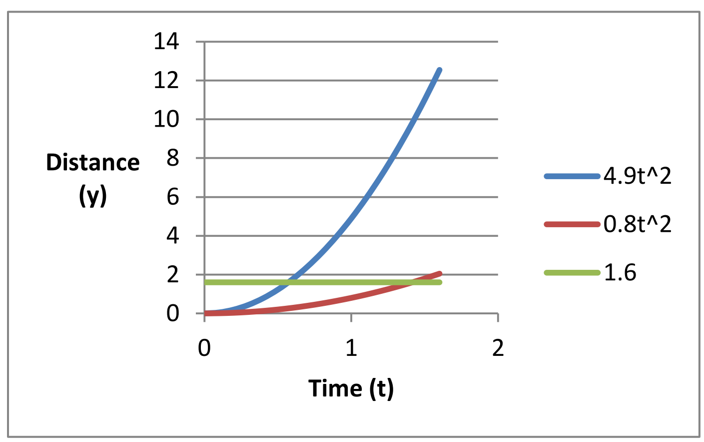
To find the speed at impact we use v = **g**t. Plugging in **g** =1.6 m/s/s and t =1.4 s we get

v(t=1.4 s) = (1.6 m/s/s)(1.4 s) = 2.3 m/s

Exercise 47B Show that it would take 0.57 s for an object to fall 1.6 m on earth. Drop a coin 1.6 m and measure the time it falls.



Velocity versus time on earth and moon (v = **g**t)



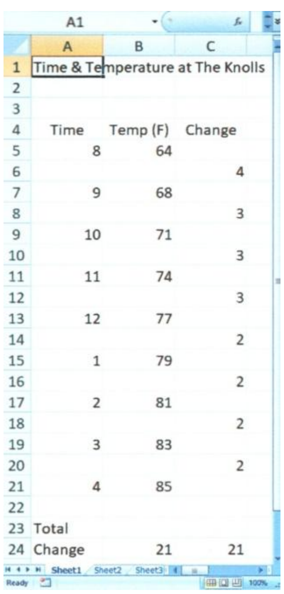
y(t) = (1/2)$\mathbf{g}t^2$ on earth and moon

UNIT 4 INTEGRAL CALCULUS

Time and Temperature at The Knolls

It's a glorious summer day at The Knolls of Oxford, a retirement community. Cumulus giants surge toward the campus. The fountain in the pond salutes them.

A Knolls resident who is learning to use spreadsheets records the temperature starting at 8 AM and ending at 4 PM. She includes the hourly changes in temperature and the total change over the 8-hour period.



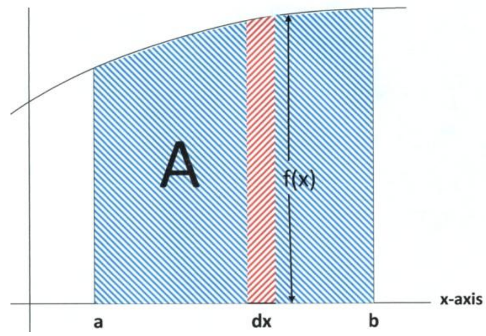
	A	B	C
1	Time & Temperature at The Knolls		
2			
3			
4	Time	Temp (F)	Change
5	8	64	
6			4
7	9	68	
8			3
9	10	71	
10			3
11	11	74	
12			3
13	12	77	
14			2
15	1	79	
16			2
17	2	81	
18			2
19	3	83	
20			2
21	4	85	
22			
23	Total		
24	Change	21	21

In column B she enters a formula that subtracts the 8AM temperature from the 4PM temperature, giving the total change (21°F). In column C she enters a formula that adds the hourly changes. The two results are equal because the total change equals the sum of the hourly changes.

What does this have to do with calculus? A **key idea** in integral calculus is that the total change in a quantity equals the sum of small changes. Hold that thought.

The Definite Integral

A basic problem in integral calculus is to determine areas. The blue area **A** under a graph of the function f(x) is shown in the figure.

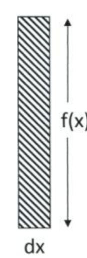


Symbols are used as shorthand in math. The symbolism used to designate the area A in the figure above is

$$\mathbf{A} = \int_a^b f(x)dx \qquad \text{The integral symbol}$$

Math Speak: **A** is called "the **integral** of f(x) from x = a to x = b". The function f(x) is called the **integrand**. **a** and **b** are called **limits**. They define the **range** of integration.

The red stripe above is a rectangle. Its height is f(x). Its width is dx. Its area is f(x)·dx.

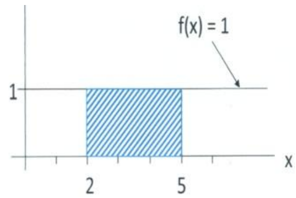


The integral sign $\int_a^b$ tells you to add the areas of the set of rectangles which "fill" the region between x = **a** and x = **b**. This gives the total area, **A**.

Integration is a summation. You are adding areas.

Let's look at an example,

$A = \int_2^5 f(x)dx = \int_2^5 1 \cdot dx$



The integrand is f(x) = 1

The quantity 1·dx is the area of a rectangle of height 1 and width dx. Adding these areas between x = 2 and x = 5 gives the area of a rectangle 1 unit high and 3 units long, A = 1 · 3 = 3.

Another interpretation of $\int_2^5 dx$ is this: "The sum of differential changes in x equals the total change in x."

$$A = \int_2^5 dx = 5 - 2 = 3$$

This is the calculus version of the Knolls temperature change story. It's an interpretation that will let us evaluate a large class of integrals. Hold the thought.

Our next quest is the "holy grail" of integral calculus – an equation that relates the integral and the derivative. We start with the Differential Relation which uses the derivative $(\frac{dy}{dx})$ to relate the differentials dx and dy

$$(\frac{dy}{dx})dx = dy \qquad \textbf{Differential Relation}$$

Form the integral of both sides. The right side becomes the **sum of changes of y** which equals the **total change of y**

$$\int_a^b (\frac{dy}{dx})dx = \int_{y(a)}^{y(b)} dy = y(b) - y(a)$$

The limits for the integral on the left [a,b] refer to values of x. The limits on the right [y(a), y(b)] refer to values of y. As x changes from a to b, y changes from y(a) to y(b). The equation

$$\int_{y(a)}^{y(b)} dy = y(b) - y(a)$$

says "the sum of differential changes in y equals the total change, y(b) – y(a)".

Let's compare the result

$$\int_a^b (\frac{dy}{dx})dx = y(b) - y(a)$$

to our original version of the integral,

$$A = \int_a^b f(x)dx$$

If we can find a function y(x) whose derivative $(\frac{dy}{dx})$ equals the integrand, f(x), we can change the form of the integral and thereby evaluate it. The original form is changed – from a sum of areas [∫ f(x) dx] to a sum of changes [∫ dy] of the function y(x),

$$\int_a^b f(x)dx = y(b) - y(a) \qquad f(x) = (\frac{dy}{dx})$$

This is it – the "holy grail" of integral calculus!

Let's review: To evaluate the integral

$$A = \int_a^b f(x)dx$$

we find a function y(x) whose derivative equals the integrand f(x)

$$(\frac{dy}{dx}) = f(x)$$

The integral becomes $_a\int^b (\frac{dy}{dx})dx$ which equals the total change in y(x)

$$\int_a^b f(x)dx = y(b) - y(a) \qquad f(x) = (\frac{dy}{dx})$$

This is cool! We can evaluate the integral if we can figure out y(x) when we know the derivative $(\frac{dy}{dx})$. Let's look at examples using Table 1.

Example. Use $\int_a^b f(x)dx$ = y(b) – y(a) to determine the area under the graph of f(x) = 2x between x = 0 and x = 5.

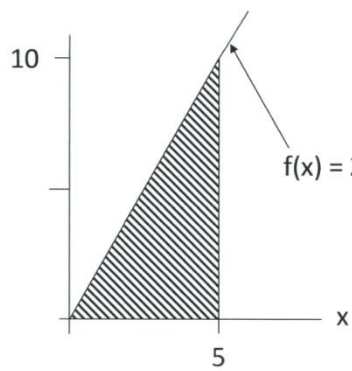


Table 1 shows

$f(x) = (\frac{dy}{dx}) = 2x$ has $y(x) = x^2$

The area equals the value of the integral

$A = \int_0^5 2x\,dx = y(5) - y(0) = 5^2 - 0^2 = 25$

We confirm this by noting that the area of the triangle, base 5, height 10, equals $\frac{1}{2}$·base·height = 25.

Example. Use $\int_a^b f(x)dx$ = y(b) – y(a) to determine the integral of $f(x) = e^x$ between x = 1 and x = 3. Table 1 shows

$f(x) = (\frac{dy}{dx}) = e^x$ has $y(x) = e^x$

The value of the integral is 17.36726.

$\int_1^3 e^x dx = y(3) - y(1) = e^3 - e^1 = 17.36726$

Example. Use $\int_a^b f(x)dx$ = y(b) – y(a) to determine the area under the graph of f(x) = cos x between x = 0 and x = π/2.

Table 1 shows

$f(x) = (\frac{dy}{dx}) = \cos x$ has $y(x) = \sin x$

The area equals the value of the integral

$\int_0^{\pi/2} \cos x\, dx = y(\pi/2) - y(0)$

$= \sin \pi/2 - \sin 0 = 1$

Example. Use $\int_a^b f(x)dx$ = y(b) – y(a) and Table 1 to evaluate

$C = \int_2^5 (\frac{1}{x})dx$

Table 1 shows

$(\frac{dy}{dx}) = \frac{1}{x}$ has $y(x) = \ln x$

so

$C = \int_2^5 (\frac{1}{x})dx = \ln 5 - \ln 2$

$= 1.609 - 0.693 = 0.916$

Exercise 48. Use $\int_a^b f(x)dx$ = y(b) – y(a) and Table 1 to show that

$D = \int_3^5 (-\sin x)dx = 1.274$

Exercise 49. Use $\int_a^b f(x)dx$ = y(b) – y(a) and Table 1 to show that

$$E = \int_2^4 (x - \sin x)\, dx$$

$$= 6 + \cos 4 - \cos 2 = 5.762$$

Exercise 50. Use $\int_a^b f(x)dx$ = y(b) – y(a) and Table 1 to evaluate the integrals

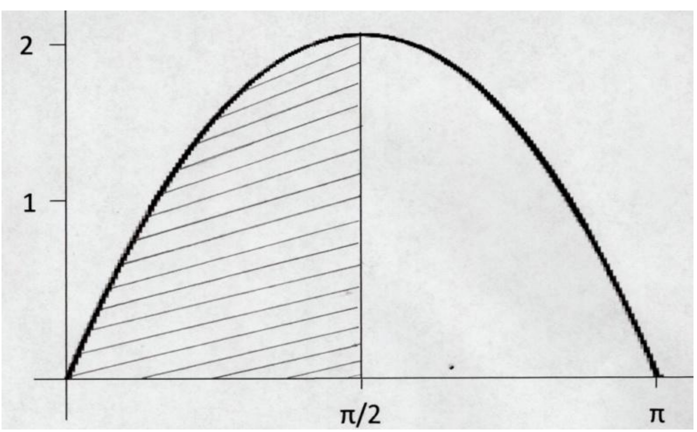
$$B = \int_3^6 x^2\, dx$$

$$D = 5\int_0^{\pi/2} \sin x\, dx$$

$$E = \int_1^2 e^{3x}\, dx$$

$$G = 3\int_0^{\pi/4} [\sec x]^2\, dx$$

Exercise 51. Which integral corresponds to the hatched area in the figure?



A) $3\int_{-\pi}^{\pi} \cos x\, dx$

B) $\int_0^{\pi/2} \sin x\, dx$

C) $\int_0^{\pi} \sin x\, dx$

D) $2\int_0^{\pi/2} \sin x\, dx$

E) $4\int_0^{\pi/4} \cos x\, dx$

Exercise 52. Use $\int_a^b f(x)dx$ = y(b) – y(a) and Table 1 to evaluate

$$D = \int_0^{\pi} \sin x\, dx$$

Exercise 53. Use $\int_a^b f(x)dx$ = y(b) – y(a) and Table 1 to evaluate the integrals

$$E = \int_2^4 x^3 dx$$

$$F = 5\int_0^2 x^4 dx$$

$$G = 6\int_0^3 x^5\, dx$$

Evaluating integrals by matching integrands with derivatives is a good way to start but there is more to the story. Let's develop the idea that the integral represents an area.

Figure 10 shows graphs of two functions, $f(x) = 9$ and $g(x) = x^2$. We want to determine the hatched area between the two graphs.

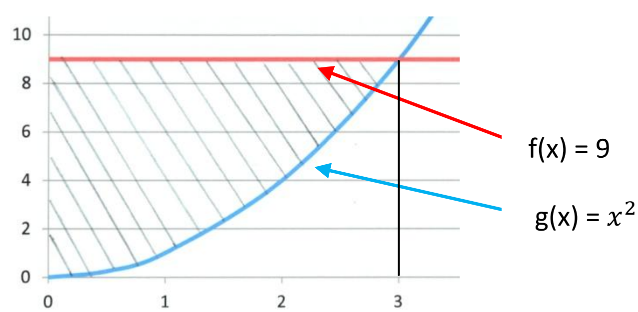


Figure 10. The hatched area is the difference of two areas.

We do this by subtracting the area beneath the parabola g(x) = x^2 from the rectangular area beneath the line f(x) = 9.

A = Area under f(x) = 9 – Area under g(x) = x^2

The areas are evaluated by integrals

A = $\int_0^3 9dx - \int_0^3 x^2 dx$ = 9(3) – ($\frac{1}{3}$)3^3 = 18

The upper limit of integration is the value of x where the two graphs intersect, f(x) = g(x), or 9 = x^2. This shows x = 3 is the upper limit.

Exercise 54. Figure 11 shows graphs of $f(x) = 6x - x^2$ and $g(x) = 2x$. Determine the hatched area between the graphs. Note: You must figure out the values of x where the graphs intersect. **These are the limits of the integration**.

Figure 11 suggests the two points are x = 0 and x = 4. Solve the equation f(x) = g(x) to prove that these are the exact values.

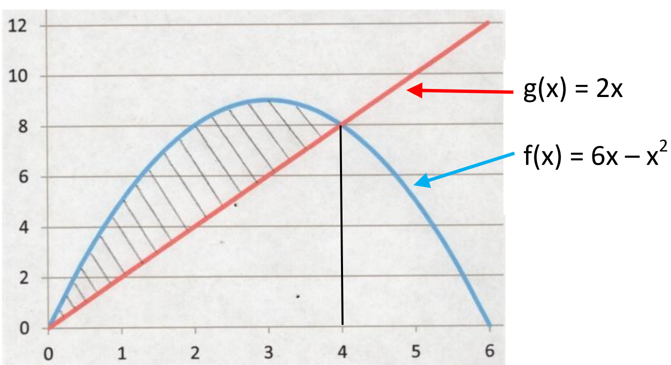


Figure 11. The hatched area is the difference of two areas.

The Indefinite Integral

We developed the integral by showing its connection to areas. Our basic result displays how the limits of integration (a,b) enter the process.

$\int_a^b (\frac{dy}{dx})dx$ = y(b) – y(a)   Definite Integral

This form is called the **definite integral** because the limits a and b define a specific range of integration. A variation, called the **indefinite integral,** emphasizes a different viewpoint. We remove the limits (a,b) and write

$\int(\frac{dy}{dx})dx = \int dy = y(x)$ Indefinite Integral

This relation shows that i**ntegrating the derivative of a function returns the original function.** It reveals **integration as the inverse of differentiation** and establishes the basic connection between integral calculus and differential calculus.

Let's look at an example. Table 1 shows that ($\frac{dy}{dx}$) = 2x is the derivative of the function y(x) = x^2. It follows that

$\int 2x\,dx = \int dy = x^2$

This is not quite the complete story. Figure 12 suggests why. Figure 12 shows graphs of three functions:

y1(x) = x^2 + 5,   y2(x) = x^2 ,   y3(x) = x^2 - 7

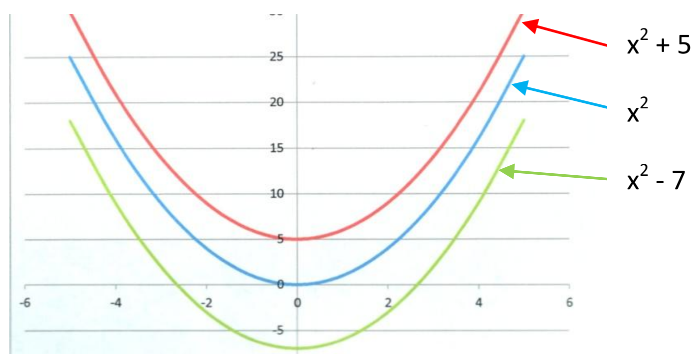


Figure 12. The graphs of three different functions which have the same derivative.

All three functions have the same derivative,

$$(\frac{dy}{dx}) = 2x$$

y(x) could be x^2 or $x^2 + 5$ or $x^2 - 7$. In fact, y(x) could be x^2 + any constant. For this reason, when you evaluate an indefinite integral you always add a constant as a reminder of this uncertainty. For the present example,

$$\int 2x\,dx = x^2 + C \qquad (C = \text{constant})$$

Math Speak: The constant C is called the "constant of integration".

One question comes to mind. Does the constant of integration affect the value of a **definite** integral? The answer is "No" because the definite integral always involves the **difference** between two values of the function.

$$\int_a^b \left(\frac{dy}{dx}\right)dx = y(b) - y(a) \quad \text{Definite Integral}$$

The constant of integration disappears when you evaluate the difference y(b) − y(a).

Example. Evaluate the indefinite integral

$$\int 4x^3\,dx$$

In order to use the basic relation

$$\int \left(\frac{dy}{dx}\right)dx = y(x)$$

we need the function whose derivative equals $4x^3$.

Do you remember the 800-pound gorilla?

$$\left(\frac{d\,x^N}{dx}\right) = N \cdot x^{N-1}$$

For N = 4 this becomes

$$\left(\frac{d\,x^4}{dx}\right) = 4x^3$$

showing that $4x^3$ is the derivative of x^4. This means

$$\int 4x^3\,dx = x^4 + C$$

Example. We can use the 800-pound gorilla to work out an equation for the integral

$$\int x^N\,dx \qquad \text{where } N = \text{constant}$$

In order to use our basic relation,

$$\int \left(\frac{dy}{dx}\right)dx = y(x)$$

we must find a function whose derivative equals the integrand, x^N.

$$\left(\frac{dy}{dx}\right) = x^N \qquad y(x) = ?$$

We start with the 800-pound gorilla relation,

$$y(x) = x^N \qquad \left(\frac{dy}{dx}\right) = N\,x^{N-1}$$

In these two equations we first replace N by N + 1. This gives

$$y(x) = x^{N+1} \qquad \left(\frac{dy}{dx}\right) = (N+1)\,x^N$$

The derivative is not quite x^N because of the factor (N+1). Divide by N+1. This leaves

$$y(x) = \frac{x^{N+1}}{(N+1)} \qquad \left(\frac{dy}{dx}\right) = x^N$$

Note that N = -1 is not allowed. Why not?

The derivative of $\frac{x^{N+1}}{(N+1)}$ equals x^N so the integral of x^N is

$$\int x^N\,dx = \frac{x^{N+1}}{(N+1)} + C \qquad N \neq -1$$

Give this result the "acid" test: The integrand must equal the derivative of the result.

$$\int f(x)\,dx = y(x) \qquad \text{"Acid" Test}$$

the integrand must equal the derivative of this

Example. Evaluate the indefinite integral

$$A = \int(x^2 - x^{-2})\,dx$$

We use $\int x^N\,dx = \frac{x^{N+1}}{(N+1)} + C$ twice

$$\int x^2\,dx = \frac{x^3}{3} \qquad \int x^{-2}\,dx = -x^{-1}$$

Add the magic constant C and we are done

$$A = \int(x^2 - x^{-2})\,dx = \frac{x^3}{3} + x^{-1} + C$$

Confirm by differentiating the answer

$$\frac{d(x^3/3 + x^{-1} + C)}{dx} = x^2 - x^{-2} + 0$$

Exercise 55. Evaluate the indefinite integral

$$B = \int(x^4 + 3x)\,dx$$

Example. Evaluate $\int \cos x\,dx$.

Table 1 shows cos x is the derivative of sin x

$$\cos x = \left(\frac{d\sin x}{dx}\right)$$

It follows that

$$\int \cos x\,dx = \int\left(\frac{d\sin x}{dx}\right)dx = \sin x + C$$

Hence,

$$\int \cos x\,dx = \sin x + C$$

Table 2 lists several indefinite integrals. C denotes the constant of integration.

$$\int A\,dx = Ax + C \qquad A = \text{constant}$$

$$\int x^N\,dx = \frac{x^{N+1}}{(N+1)} + C \qquad N \neq -1$$

$$\int \frac{dx}{x} = \ln x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$$

Table 2. Integrals of selected functions

Exercise 56. Confirm that

$$-\cos x + C = \int \sin x\,dx$$

by subjecting it to the acid test.

Exercise 57. Evaluate the following indefinite integrals. Check your answers by using the "acid" test.

$A = \int(x^2 + x + 6)\, dx$

$B = \int(x^{-2} - x^2)\, dx$

$D = \int (e^{2x} + 1)\, dx$

Example. There is one exception to the equation we worked out for the integral of x^N. The value $N = -1$ is not allowed because of the factor of N + 1 in the denominator. Let's evaluate $\int x^{-1}\, dx = \int \frac{dx}{x}$. Our basic relation, $\int(\frac{dy}{dx})dx = y(x)$ says that the integral of the derivative equals the original function. In Unit 3 we showed that $\frac{1}{x}$ is the derivative of ln x. This means that the integral of $\frac{1}{x}$ equals $\ln x + C$,

$$\int x^{-1}\, dx = \int \frac{dx}{x} = \ln x + C$$

The Substitution Method for Integrals

Some types of integrals can be evaluated by making a substitution that changes the variable of integration. For example, consider

$$S = \int 2x\, (x^2 + 1)^4\, dx$$

Make the substitution $z = (x^2 + 1)$ and note that $(\frac{dz}{dx}) = \frac{d\,(x^2+1)}{dx} = 2x$. Using the differential relation

$$dz = (\frac{dz}{dx})dx = 2x\, dx$$

transforms the integral

$$S = \int 2x\, (x^2 + 1)^4\, dx = \int z^4\, dz$$

The variable of integration in the starting form is x. The variable of integration changes to z as a result of the substitutions. The form of the integrand, z^4, is more familiar than the original form. We can evaluate it with the help of an entry in Table 2.

$$\frac{z^{N+1}}{(N+1)} + C = \int z^N\, dz$$

With N = 4 this becomes

$$\frac{z^5}{5} + C = \int z^4\, dz$$

Using this we get

$$\int 2x\, (x^2 + 1)^4\, dx = \int z^4\, dz = \frac{1}{5} z^5 + C = \frac{1}{5} (x^2 + 1)^5 + C$$

Once we carry out the z integration we replace z by $(x^2 + 1)$.

The key steps: 1) identify the substitution, $z = (x^2 + 1)$ and 2) change the variable of integration, $dz = (\frac{dz}{dx})dx = 2x\, dx$.

Example. Evaluate $T = \int(\sin x)^4 \cos x\, dx$

For our substitution we choose $z = \sin x$

To change the variable of integration from x to z we remember that

$$\cos x = (\frac{d \sin x}{dx})$$

The differential relation gives

$$dz = \left(\frac{dz}{dx}\right)dx \quad => \quad dz = \cos x\, dx$$

The form of T is changed to the same form encountered above with the integral S,

$$T = \int[\sin x]^4 \cos x\, dx = \int z^4\, dz$$

It follows from $\int z^N\, dz = \frac{z^{N+1}}{(N+1)} + C$

that

$$T = \int z^4\, dz = \frac{1}{5} z^5 + C = \frac{1}{5}[\sin x]^5 + C$$

The acid test: Make sure the derivative of the answer equals the integrand.

Exercise 58. Evaluate the following indefinite integrals by making a substitution. Check your answers by using the “acid” test. The chain rule may help with the acid test.

$F = \int 2x\, e^{x^2}\, dx$

$G = \int 12x^2(4x^3 + 5)^{3/2}\, dx$

$H = \int(1 - \frac{1}{x}) \cos(x - \ln x)\, dx$

Example. Given that $\int_1^9 f(u)du = 6$

Prove that $\int_1^3 8x \cdot f(x^2)dx = 24$

Let $x^2 = u$. Then $2xdx = du$ and the integral can be written as

$$4\int_1^3 2x \cdot f(x^2)dx = 4\int_1^9 f(u)du = 4 \cdot 6 = 24$$

Notice that the limits of integration for x and u are different. The limits for x must be squared to give the limits for u.

You try this one: Given $\int_{11}^{17} f(u)du = 5$

Evaluate $\int_3^6 4xf(2x+5)dx$

and this one: Given $\int_7^{34} f(u)du = 15$

Evaluate $\int_0^3 3x^2 f(x^3 + 7)dx$

Numerical Integration

Many integrals cannot be evaluated analytically. For example, the study of ellipses leads to so-called elliptic integrals. One form of elliptic integral is

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k \cdot [\sin x]^2}\, dx$$

There is no function y(x) whose derivative equals

$$\sqrt{1 - k \cdot [\sin x]^2}$$

Elliptic integrals are evaluated numerically using computer programs that add areas.

Exercise E. The elliptic integral E(k) can be evaluated analytically for two special values of k. Show that $E(0) = \frac{\pi}{2}$ and $E(1) = 1$.

UNIT 5 EULER FEST

Infinity is a journey, not a destination.

Our journey is nearing its end. We have visited derivatives and integrals. This final unit takes another look at both and adds a bonus – a magical relation known as Euler's Formula.

Euler's Number

Euler's number, symbolized as **e,** is named for the Swiss mathematician Leonhard Euler (pronounced "Oiler"). It appears in many areas of math and science. Euler's number is irrational. An approximate value is e ≈ 2.718281828459045.

Euler's number can be expressed as a value of the function $\boldsymbol{e^x}$ defined by the sum

$$e^x = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} + \cdots +$$

The **factorial** (n!) is defined by a start value (0! = 1) and a **recursion relation** (n! = n·(n-1)!)

0! = 1 n! = n·(n-1)! => 1! = 1 2! = 2 3! = 6 4! = 24 5! = 120

Exercise 59. Evalulate A) 6! B) 7! C) 8! D) 9!

Euler and others proved that **e** is irrational. The series which defines **e** is composed of rational numbers. It can **approach** but not reach the irrational limit value. Setting x = 1 in the series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ lets us evaluate Euler's number.

$$\boldsymbol{e} = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots$$

Successive terms get smaller and smaller. The first few give a reasonably accurate value for **e**.

Example. If you use a calculator to evaluate the first 6 terms in the sum you get

e ≈ 1 + 1 + ½ + 1/6 + 1/24 + 1/120 = 2.71667

Exercise 60. How many terms in the sum for **e** are needed to give the value 2.718281826 ?

We can use the series definition to prove a unique property of the function e^x, namely that its derivative equals the function,

$$\left(\frac{d\,e^x}{dx}\right) = e^x$$

Differentiating the series term by term gives [use $\left(\frac{d\,x^N}{dx}\right) = Nx^{N-1}$ and $\frac{k}{k!} = \frac{1}{(k-1)!}$]

$$\left(\frac{d\,e^x}{dx}\right) = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \frac{5x^4}{120} + \frac{6x^5}{720} + \cdots + \frac{kx^{k-1}}{k!} + \frac{(k+1)x^k}{(k+1)!} + \cdots$$

$$\left(\frac{d\,e^x}{dx}\right) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots + \frac{x^{k-1}}{(k-1)!} + \frac{x^k}{k!} + \cdots$$

The derivative of the series equals the series. This shows that $\left(\frac{d\,e^x}{dx}\right) = e^x$

Exercise 61. For very small values of x the exponential function can be approximated by the first two terms in the series

$$e^x \approx 1 + x \qquad x << 1$$

Use this approximation and the identity $\ln(e^x) = x$ to show that

$$\ln(1 + x) \approx x$$

This result was used earlier to derive the Rule of 72.

Exercise 62. The approximation $\ln(1 + \frac{r}{100}) \approx \frac{r}{100}$ was used to derive the Rule of 72, where r is the interest rate (in %). Typically, r is less than 10. Use a calculator to compare $\ln(1 + \frac{r}{100})$ and $\frac{r}{100}$ for r = 3, 6 and 10. Would you say the approximation is a good one?

Euler's Formula

Euler's formula relates the exponential function e^{ix} and the trigonometric sine and cosine functions, sin x and cos x. The quantity **i** denotes the square root of -1 ($i^2 = -1$).

e^{ix} = cos x + i sin x   Euler's Formula

There are many mystical relations in mathematics. Euler's formula is magical!

The most famous instance of Euler's formula sets $x = \pi$ (π radians <=> 180°) for which

$$\cos \pi = -1 \quad \text{and} \quad \sin \pi = 0$$

Euler's formula becomes

$$e^{i\pi} = \cos \pi + i \sin \pi = -1$$

or

$$\mathbf{e^{i\pi} + 1 = 0}$$

thereby connecting seven of the most famous symbols in mathematics, e, i, π, 1, +, = and 0.

We want to prove Euler's formula. The proof starts by recalling that the derivative of a constant is zero. **Let's turn that around**. If we have a function Q(x) whose derivative is zero for all values of x we know that the function is a constant. Symbolically,

$$\frac{dQ}{dx} = 0 \quad => \quad Q(x) = \text{constant}$$

The function Q(x) we deal with is a quotient

$$Q(x) = \frac{f(x)}{g(x)} = \frac{\cos x + i \sin x}{e^{ix}}$$

The idea behind the proof is to show that the quotient of f(x) = cos x + i sin x and g(x) = e^{ix} is a **constant** and **equal to one**. This will prove f(x) and g(x) are equal, i.e. it will prove Euler's formula.

The rule for the derivative of the quotient of two functions is

$$\frac{d[\frac{f(x)}{g(x)}]}{dx} = \frac{g(x)(\frac{df}{dx}) - f(x)(\frac{dg}{dx})}{[g(x)]^2} \qquad \text{Quotient Rule}$$

A proof of this rule is given at the end of this unit.

A reminder of where we are heading: We want to prove that the derivative of the quotient $\frac{f(x)}{g(x)}$ is zero. We do this by showing that the numerator $g(x)(\frac{df}{dx}) - f(x)(\frac{dg}{dx})$ is zero. To do this we need the derivatives $(\frac{df}{dx})$ and $(\frac{dg}{dx})$. From Table 1 we see that the derivative of cos x is – sin x and the derivative of sin x is cos x. For

f(x) = cos x + i sin x

the derivative becomes

$$(\frac{df}{dx}) = -\sin x + i \cos x$$

Using $-1 = i^2$ the right side can be expressed as i(cos x + i sin x). This maneuver gives us one of the derivatives we need

$(\frac{df}{dx})$ = i (cos x + i sin x) = i f(x)

Our study of the exponential function produced the result

$$(\frac{d\,e^{kx}}{dx}) = k \cdot e^{kx}$$

Using this with k = i for g(x) = e^{ix} gives

$$(\frac{dg}{dx}) = (\frac{d\,e^{ix}}{dx}) = i\,e^{ix} = i\,g(x)$$

Plugging these results for $(\frac{df}{dx})$ and $(\frac{dg}{dx})$ into the quotient rule proves that the derivative of the quotient is zero

$$\frac{d[\frac{f(x)}{g(x)}]}{dx} = \frac{g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}}{[g(x)]^2} = \frac{g(x)\cdot i\,f(x) - f(x)\cdot i\,g(x)}{[g(x)]^2} = 0$$

We have shown that the derivative of the quotient is zero for all values of x. This proves that the quotient is a constant. The value of the constant equals 1. We prove this by evaluating the quotient for x = 0

f(0) = cos 0 + i sin 0 = 1 + i 0 = 1   g(0) = e^{i0} = e^0 = 1

Both f(0) and g(0) equal one, so their quotient equals one for all values of x.

$$\frac{f(x)}{g(x)} = \frac{f(0)}{g(0)} = 1 \quad => \quad g(x) = f(x)$$

This completes the proof of Euler's formula

e^{ix} = cos x + i sin x   Euler's Formula

Example. Euler's formula can be used to work out the values of quantities like $\sqrt{i}$ and $\sqrt[3]{i}$. Recall that fractional exponents are used to represent roots, for example

$$\sqrt{z} = z^{1/2}$$

Note that $\frac{\pi}{2}$ radians <=> 90° $\frac{\pi}{4}$ radians <=> 45°

You need values of the sines and cosines,

$$\sin\frac{\pi}{2} = 1 \qquad \cos\frac{\pi}{2} = 0$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

In Euler's formula set x = $\frac{\pi}{2}$ to show

$$e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = i$$

To find the square root of **i** you use

$$\sqrt{i} = i^{1/2} = (e^{i\pi/2})^{1/2} = e^{i\pi/4}$$

Then use Euler's formula for $e^{i\pi/4}$

$$\sqrt{\boldsymbol{i}} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\mathbf{1+i}}{\sqrt{\mathbf{2}}}$$

Confirm by squaring both sides

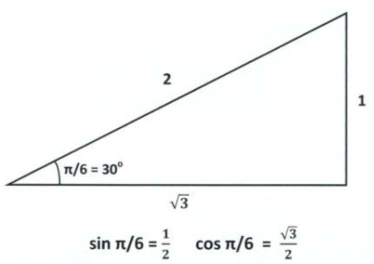
$$i = \sqrt{i}\sqrt{i} = \left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}}\right) = \frac{1^2 + 2i + i^2}{2} = i$$

For the cube root of **i** you start with

$$\sqrt[3]{i} = i^{1/3} = (e^{i\pi/2})^{1/3} = e^{i\pi/6}$$

and then use Euler's formula to express $e^{i\pi/6}$ in terms of sin $\frac{\pi}{6}$ and cos $\frac{\pi}{6}$

Exercise 63. Carry out the procedure outlined above to show that



$$\sqrt[3]{i} = \frac{\sqrt{3}+i}{2}$$

Confirm by showing that $\left(\frac{\sqrt{3}+i}{2}\right)^3 = i$

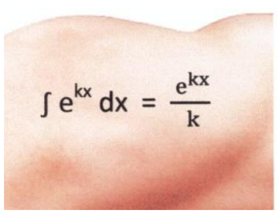
Exercise 64. If you replace i by – i in Euler's formula you get

$$e^{-ix} = \cos x - i \sin x$$

Multiply this form by $e^{ix} = \cos x + i \sin x$ to prove the Pythagorean identity for sines and cosines,

$$[\cos x]^2 + [\sin x]^2 = 1$$

A Final Example. Ted was not fond of trigonometry. He avoided sines and cosines whenever possible. Ted loved exponential functions like e^{kx} and e^{-kx}. He had tattoos showing $\int e^{kx}\,dx = \frac{e^{kx}}{k}$ and $\left(\frac{d\,e^{kx}}{dx}\right) = k{\cdot}e^{kx}$



Ted realized he could evaluate the integrals of sin x and cos x by using Euler's formula.

He started by integrating Euler's formula

$$\int e^{ix}\,dx = \int \cos x\,dx + i \int \sin x\,dx$$

Next, Ted checked his tattoo and wrote

$$\int e^{ix}\,dx = \frac{e^{ix}}{i}$$

He replaced e^{ix} by cos x + i sin x and ended up with a second equation for $\int e^{ix}\,dx$

$$\int e^{ix}\,dx = \frac{e^{ix}}{i} = \frac{\cos x + i \sin x}{i} = \sin x - i \cos x$$

If two complex numbers are equal their real parts must be equal and their imaginary parts must be equal*. [* U + iV = Y + iZ requires U = Y and V = Z]

Comparing the two expressions for $\int e^{ix}\,dx$ let Ted write

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

Note that Ted was careful to add the obligatory constant of integration. He celebrated by having T-shirts made that displayed $e^{i\pi} + 1 = 0$

$e^{i\pi}$ Club 2019

Exercise 65. We have confirmed every entry in Table 1 except one: $\left(\frac{d\tan x}{dx}\right)$ = $[\sec x]^2$
Prove this by using the quotient rule,

$$\frac{d[\frac{f(x)}{g(x)}]}{dx} = \frac{g(x)(\frac{df}{dx}) - f(x)(\frac{dg}{dx})}{[g(x)]^2} \quad \text{Quotient Rule}$$

Start with the identity $\tan x = \frac{\sin x}{\cos x}$

Use the quotient rule with f(x) = sin x and g(x) = cos x. You will need to recognize the identity $[\sin x]^2 + [\cos x]^2 = 1$ and remember that $\sec x = \frac{1}{\cos x}$

Exercise 66. A) Start with Euler's formula and the series definition of e^{ix}

$$\cos x + i\sin x = e^{ix} = \sum_{n=0}^{\infty}\frac{(ix)^n}{n!}$$

$$= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{6} + \frac{(ix)^4}{24} + \frac{(ix)^5}{120} + ..$$

Derive the first three terms in the series formulas for sin x and cos x

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} \;\ldots\ldots$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} \;\ldots\ldots.$$

B) The derivative of sin x equals cos x. Show that the derivative of the series for sin x equals the series for cos x.

Exercise 67. Use Euler's formula to show that the 14th root of **i** is given by

$i^{1/14} = 0.9937 + i(0.1120)$

Exercise 68. Start with the relation

$$\left(\frac{d\, e^{ikx}}{dx}\right) = ik \cdot e^{ikx} \qquad [\text{ k = constant }]$$

A) Use Euler's formula to show that the derivatives of sin kx and cos kx are given by

$$\left(\frac{d\sin kx}{dx}\right) = k \cdot \cos kx$$

$$\left(\frac{d\cos kx}{dx}\right) = -\, k \cdot \sin kx$$

Remember U + iV = Y + iZ requires

U = Y and V = Z

B) Show that the case k = 1 produces the results listed in Table 1.

Product, Quotient, Chain Rules and Trigonometric Addition Relations

We used several rules and relations along our journey. Here are brief derivations of four.

The Product Rule relates the derivative of the product of two functions to the derivatives of the individual functions.

$$\left(\frac{d[f(x)g(x)]}{dx}\right) = f(x)\left(\frac{dg}{dx}\right) + g(x)\left(\frac{df}{dx}\right)$$

Start with the definition

Subtract and add f(x+h)g(x) and then factor

$$\left(\frac{d[f(x)g(x)]}{dx}\right) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0}\frac{g(x)\,[f(x+h) - f(x)]}{h} = f(x)\left(\frac{dg}{dx}\right) + g(x)\left(\frac{df}{dx}\right)$$

Recognize the limits f(x) $\left(\frac{dg}{dx}\right)$ and $\left(\frac{df}{dx}\right)$

The Quotient Rule is a special case of the product rule. In the product rule, replace g(x) by

$$\frac{1}{g(x)} = g(x)^{-1}$$

$$\frac{d[\frac{f(x)}{g(x)}]}{dx} = \left(\frac{d[f(x)\,g(x)^{-1}]}{dx}\right) = f(x)\left(\frac{d\,g^{-1}}{dx}\right) + g(x)^{-1}\left(\frac{df}{dx}\right)$$

By differentiating $1 = \frac{g(x)}{g(x)} = g(x)\cdot g^{-1}(x)$ we can show $\left(\frac{d\,g^{-1}}{dx}\right) = -g^{-2}\left(\frac{dg}{dx}\right)$

$$\frac{d[\frac{f(x)}{g(x)}]}{dx} = f(x)\left[-g(x)^{-2}\left(\frac{dg}{dx}\right)\right] + g(x)^{-1}\left(\frac{df}{dx}\right) = \frac{g(x)(\frac{df}{dx}) - f(x)(\frac{dg}{dx})}{[g(x)]^2}$$

Quotient Rule

Chain Rule $\left(\frac{df}{dx}\right) = \lim_{h\to 0}\frac{f(z(x+h)) - f(z(x))}{h} = \lim_{h\to 0}\frac{[f(z(x+h)) - f(z(x))]\cdot[z(x+h) - z(x)]}{h\cdot[z(x+h) - z(x)]} = \lim_{h\to 0}\frac{[f(z(x+h)) - f(z(x))]}{[z(x+h) - z(x)]}\,\frac{[z(x+h) - z(x)]}{h} = \left(\frac{df}{dz}\right)\left(\frac{dz}{dx}\right)$

Insert $[z(x+h) - z(x)]$ in numerator and denominator   move pieces a tad   recognize limits as derivatives

Trigonometric Addition Relations $e^{ix}\,e^{ih} = e^{i(x+h)}$ Use Euler's formula on both sides

$e^{ix}\,e^{ih}$ = (cos x + i sin x)(cos h + i sin h) = cos x cos h – sin x sin h + **i**(sin x cos h + cos x sin h)

$e^{i(x+h)}$ = cos (x+h) + **i** sin (x+h) => cos (x+h) = cos x cos h – sin x sin h & sin (x+h) = sin x cos h + cos x sin h

Appendix 1. Pascal's Triangle

The binomial expansion is used frequently in algebra and calculus. Here are examples:

$(x + a)^0 = 1$

$(x + a)^1 = x + a$

$(x + a)^2 = (x + a)(x + a) = x^2 + 2xa + a^2$

$(x + a)^3 = (x + a)(x + a)(x + a) = x^3 + 3x^2a + 3xa^2 + a^3$

$(x + a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$

$(x + a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5$

The French scientist Blaise Pascal (1623-1662) observed that the coefficients of the terms can be displayed in triangular form. Pascal's triangle makes it easy to write down the coefficients for any power of (x + a). The rule:

Each coefficient is the sum of the two nearest coefficients in the line above it.

PASCAL'S TRIANGLE

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 1

1 1

1 10 45 120 210 252 210 120 45 10 1

Exercise 69. Fill in the missing coefficients in the Pascal triangle above.

Exercise 70. Write out the expansion of

$(x + a)^{11}$

Appendix 2. Exercise Answers and Solutions

3 Slope = $\frac{3}{2}$

4 y = -2 at x = 2 , Yes

5 Slope = 2, b = -2

6 y = 2 x – 2   m = 2   b = -2   Yes

7 Slope = 0, 0

8 m = $\frac{1}{2}$, b = 2

9 A) $y = \frac{1}{4}x - 2$ B) y = -2 C) $y = -\frac{1}{2}x - 2$

10 B

11 2, 4, $2^{1/2}$, 32, 2^{12} = 4096, 2^0 = 1, 2^0 = 1

12

Number	Base 2	ln	log
8	3	2.079	0.903
64	6	4.159	1.806
4096	12	8.318	3.612

13 x = 3

14 A) 10^6 B) 5 C) 10

15 -7 -1 0 2

-3 $\frac{1}{5}$ 4 5

16 C ($\sqrt{x+h}$)

17 A (sin(x+h))

18 $f(x) = 4x^3 + 0.55$ A) f(3) = 108.55 B) $f(z^2) = 4z^6 + 0.55$

19 15 $A(r) = \pi r^2$

20 D (y(3) = 13)

21 $\frac{5x+2}{3x+8} = 1 \Rightarrow 5x + 2 = 3x + 8 \Rightarrow x = 3$

22 A) 12 B) 19 C) -4

23 A) 4 B) 2x C) 2 D) 1

25 A) $(\frac{d\ 10 \cdot x^3}{dx}) = 10 \cdot (\frac{d\ x^3}{dx}) = 30x^2$ B) $20e^{4x}$

26 A) $\frac{d\ (5x^2 + 2x + 7)}{dx} = \frac{d\ (5x^2)}{dx} + \frac{d(2x)}{dx} + \frac{d(7)}{dx}$
= 10x + 2 + 0 B) $13x^{12} + 7x^6 - 7x^{-8}$

27 $f \cdot g = x^2 \cdot x^2 = \frac{d(fg)}{dx} = x^2 \frac{d(\ x^2)}{dx} + \frac{d(\ x^2)}{dx} x^2 = 4x^3$

28 $(\frac{dy}{dx}) = 6 - 2x = 0 \Rightarrow$ maximum at x = 3

29 $(\frac{dy}{dx}) = x^2 - x - 6 = (x-3)(x+2) = 0$

gives valley/min at x = 3, peak/max at x = -2

30 $(\frac{dA}{dx}) = 900 - 2x = 0$ max A = 225·450

31 tangent line: y = 6x - 9

32 $(\frac{dg}{dx}) = 2x - 32x^{-3} = 0 \Rightarrow x = 2$ g(2) = 8

33 $z = (x^2 + 6x)$ $(\frac{dz}{dx}) = 2x + 6$

$\frac{d\ (z^4)}{dx} = 4z^3\ (\frac{dz}{dx}) = 4(2x+6)(x^2+6x)^3$

34 A) $15(6x-5)(3x^2-5x)^{14}$ B) $4x^3 \cos x^4$

C) $3(4x^3 + 6x^2)(x^4 + 2x^3 + 5)^2$

35 CALCULUS RULES

36 exponential B, C, E linear A, D

37 18 years 38 10 days 39 1/8 40 964 yr

41 A linear B, D Decay C, E Growth

42 $(\frac{dg}{dx}) = -3x^{-2} + x^{-1} = 0 \Rightarrow x = 3$

Minimum = g(3) = 1 + ln 3 = 2.0986

H 6 days T 1/16

43 $8x^7$ $7x^6$ $6x^5$ $5x^4$

44 $(\frac{dy}{dx}) = (\frac{d\ln x}{dx})$ $(\frac{dx}{dy}) = e^y = x$

45 A) $dy = (\frac{dy}{dx})\, dx = (\frac{d\ln x}{dx})dx = \frac{1}{x}dx$

46 $y(x) = x^3$ $(\frac{dy}{dx}) = 5$

47 x = 15 maximizes P(x). The maximum value of P(x) is (50 + 15)(800 – 10·15) = 42,250

48 $\int_3^5 \sin x\, dx = -\cos 5 + \cos 3 = -1.274$

49 $E = \int_2^4 \{x - \sin x\}dx$ $\int_2^4 x\, dx = (1/2)(4^2 - 2^2) = 6$

$\int_2^4 - \sin x\, dx = \cos 4 - \cos 2 = -0.654 - (-0.416)$

E = 6 – 0.654 + 0.416 = 5.762

50 $B = \int_3^6 x^2\, dx = (1/3)[6^3 - 3^3] = 63$

$D = 5\int_0^{\pi/2} \sin x\, dx = 5[-\cos \pi/2 + \cos 0] = 5$

$E = \int_1^2 e^{3x}\, dx = \frac{1}{3}(e^6 - e^3) = 127.78$

$G = 3\int_0^{\pi/4} \sec^2 x\, dx = 3[\tan \pi/4 - \tan 0] = 3$

51 D $2\int_0^{\pi/2} \sin x\, dx$ [Note limits, max value of integrand]

52 $D = \int_0^{\pi} \sin x\, dx = -\cos \pi + \cos 0 = 2$

53 $E = \int_2^4 x^3 dx = (1/4)[4^4 - 2^4] = 60$

$F = 5\int_0^2 x^4 dx = 5[(1/5)2^5] = 32$

$G = 6\int_0^3 x^5\, dx = 6[(1/6)3^6] = 729$

54 Area $= 4(1/2)4^2 - (1/3)4^3 = 32 - \frac{64}{3} = 10.6667$

56 $B = \frac{1}{5}x^5 + \frac{3}{2}x^2 + C$

57 A) $\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x + C$

B) $-x^{-1} - \frac{1}{3}x^3 + C$ D) $\frac{1}{2}e^{2x} + x + C$

58 $F = e^{x^2} + C$ $G = \frac{2}{5}(4x^3 + 5)^{5/2} + C$

H = sin(x – ln x) + C

59 6! = 720 7! = 5040 8! = 40320 9! = 362880

60 Start with $x = \ln(e^x)$,use the approximation $e^x \approx 1 + x$. This gives $x \approx \ln(1 + x)$

61 ln(1.03) = 0.296 ln(1.06) = 0.583) ln(1.10) = 0.953

62 12 terms gives 2.718281826

63 $(\frac{\sqrt{3}+i}{2})^3 = [\frac{(\sqrt{3})^3 + 3(\sqrt{3})^2 i + 3(\sqrt{3})i^2 + i^3}{8}]$

$= [\frac{3(\sqrt{3}) + 9i - 3(\sqrt{3}) - i}{8}] = \frac{8i}{8} = i$

64 $e^{ix}e^{-ix} = e^0 = (\cos x + i \sin x)(\cos x - i \sin x)$

$e^0 = 1 = (\cos x)^2 - i^2 (\sin x)^2 = (\cos x)^2 + (\sin x)^2$

65 $\frac{g(x)(\frac{df}{dx}) - f(x)(\frac{dg}{dx})}{[g(x)]^2} = \frac{\cos x(\cos x) - \sin x(-\sin x)}{[\cos x]^2}$

$= \frac{1}{[\cos x]^2} = [\sec x]^2$

66 $\cos x + i \sin x = e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$

$= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{6} + \frac{(ix)^4}{24} + \frac{(ix)^5}{120} + ..$

$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + i\left[x - \frac{x^3}{6} + \frac{x^5}{120}\right]....$

67 $\sqrt[14]{i} = i^{1/14} = (e^{i\pi/2})^{1/14} = e^{i\pi/28}$

$= \cos \pi/28 + i \sin \pi/28 = 0.9937 + i(0.1120)$

69 8 28 56 70 56 28 8

9 36 84 126 126 84 36 9

70 $(x + a)^{11} = x^{11} + 11x^{10}a + 55x^9a^2 + 165x^8a^3 + 330x^7a^4 + 462x^6a^5 + 462x^5a^6 + 330x^4a^7 + 165x^3a^8 + 55x^2a^9 + 11xa^{10} + a^{11}$

y(x)	$(\frac{dy}{dx})$	y(x)	$(\frac{dy}{dx})$
$m \cdot x + b$	m	e^x	e^x
x^2	$2x$	e^{kx}	$k \cdot e^{kx}$
x^3	$3x^2$	$e^{f(x)}$	$e^{f(x)}(\frac{df}{dx})$
x^N	$N\,x^{N-1}$	sin x	cos x
ln x	$\frac{1}{x}$	cos x	- sin x
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	tan x	$[\sec x]^2$

Table 1. Derivatives of selected functions

$\int A\,dx = Ax + C$ A = constant

$\int x^N\,dx = \frac{x^{N+1}}{(N+1)} + C$ $N \neq -1$

$\int \frac{dx}{x} = \ln x + C$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$

Table 2. Integrals of selected functions

Relation	**Example**
$\text{Number} = \text{Base}^{\text{Logarithm}}$	$8 = 2^3$ $1000 = 10^3$
$\log(AB) = \log A + \log B$	$\log 6 = \log 3 + \log 2$
$\log A^N = N \cdot \log A$	$\log 5^3 = 3\log 5$
$\log(\frac{A}{B}) = \log A - \log B$	$\log(\frac{6}{10}) = \log 6 - \log 10$
$\log 1 = 0$	

Logarithm Relations (All Bases)