Directions: Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Problem 1 (25 pts).

a. (5 pts) Write down the limit definition of the derivative of f at the point $x = a$ ("The most important definition in calculus", according to Long).

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

b. (8 pts) What problems can prevent a function from having a derivative at a point? (The more the merrier; figures are encouraged.)

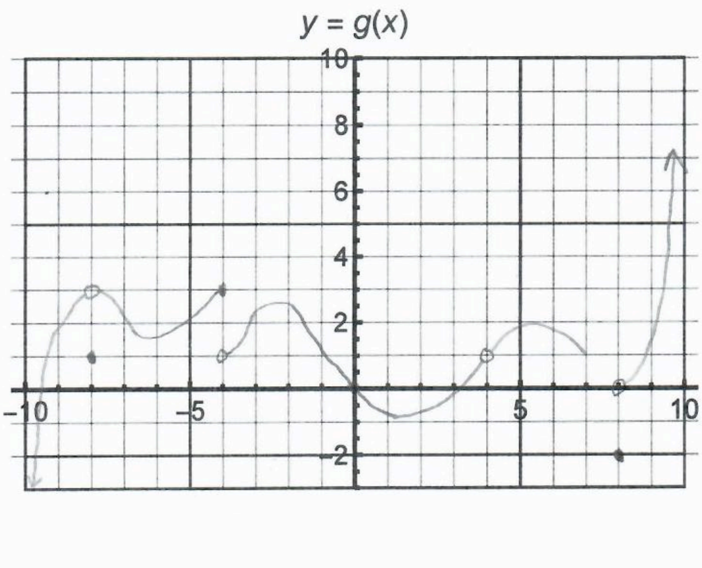
The point is on a corner such as the point (0,0) of the graph of the function $y = |x|$.

The function is not defined at that point. $f(0) = DNE$

There is a break in the function at that point

great

c. (12 pts) Draw the graph of a function g consistent with the following:



$g(-8) = 1$
$g(-4) = 3$
$g(0) = 0$
$g(4) = DNE$
$g(8) = -2$

$\lim_{x\to -8} g(x) = 3$
$\lim_{x\to -4^-} g(x) = 3$
$\lim_{x\to -4^+} g(x) = 1$
$\lim_{x\to 0} g(x) = 0$
$\lim_{x\to 4} g(x) = 1$
$\lim_{x\to 8^-} g(x) = DNE$
$\lim_{x\to 8^+} g(x) = 0$

Problem 2 (25 pts). Consider the function $f(x) = (x-1)^2 - 2$.

a. (10 pts) Use the limit definition to compute the derivative $f'(2)$.

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-1)^2 - 2 - [(x-1)^2 - 2]}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) + 1 - 2 - [x^2 - 2x + 1 - 2]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h + \cancel{1} - \cancel{2} - \cancel{x^2} + \cancel{2x} + \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h}$$

$$= \lim_{h \to 0} [2x + h - 2] = 2x - 2.$$

$$\therefore f'(2) = 2 \times 2 - 2 = 2.$$

b. (5 pts) Write the equation of the tangent line to the function at $(2, f(2))$. $\rightarrow (\cancel{x_1}2, \cancel{y_1}-1)$ $\cancel{(1, -2)}$

The point of tangent line is $\cancel{(2+h, f(2+h))}$

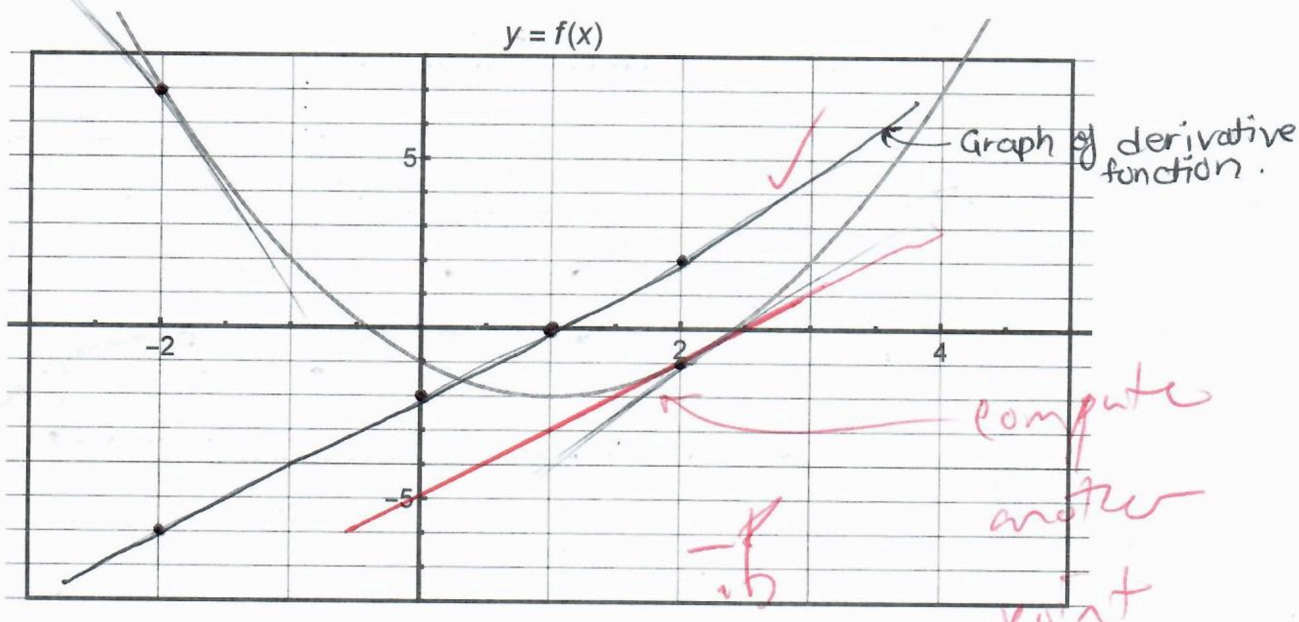
$$m = \cancel{\frac{f(2+h) - f(2)}{\cancel{2}+h-\cancel{2}} = \frac{f(2+h)-f(2)}{h}}$$

$$\cancel{y - f(2) = \frac{f(2+h) - f(2)}{h}} \cancel{(x-2)}$$

The equation of tangent line is:-

$$\therefore y + 1 = 2(x - 2)$$

c. (10 pts) Then carefully add the graph of the derivative **function**, the tangent line at $x = 2$, and the tangent line at $x = -2$ to the plot below.



compute another point (e.g. y-intercept)

Problem 3 (25 pts). Water drains from a tank over time (in minutes); here are heights (in meters):

time t (minutes)	0	1	2	3	4
height $H(t)$ (meters)	4	2	1	0.5	0.25
average rate of change	NA	-1	-.5	-.25	-.125

how?

a. (8 pts) Compute the average rate of change over each minute, and add your rates to the table above (where each answer in the box represents the average rate of change for that minute). What are the units?

meters per min

b. (5 pts) Using the data in the table above, choose two points to create a secant line whose slope approximates the derivative of H at $t = 2$. Write the equation of the secant line joining those two points. Why did you choose these two points?

(1,2) (3, 0.5)

$$\frac{.5-2}{3-1} = \frac{-1.5}{2} = -.75$$

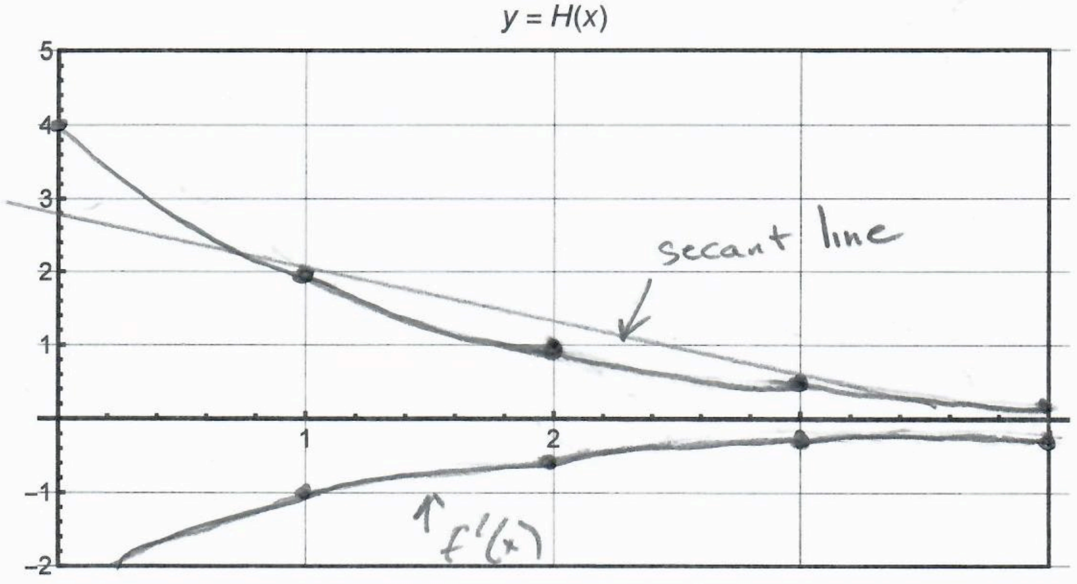
$$y-2 = -.75(x-1)$$

$$y-2 = -.75x + .75$$

$$+2 \qquad + 2$$

equation — $\boxed{y = -.75x + 2.75}$

c. (12 pts) Graph the data from the table, the secant line you used, and then your guess for the derivative function $H'(t)$.



good!

Problem 3 (25 pts). Water drains from a tank over time (in minutes); here are heights (in meters):

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height $H(t)$ (meters)	4	2	1	0.5	0.25
average rate of change	NA	-2	-1	-0.5	-0.25

a. (8 pts) Compute the average rate of change over each minute, and add your rates to the table above (where each answer in the box represents the average rate of change for that minute). What are the units?

Units = meters per minutes.

b. (5 pts) Using the data in the table above, choose two points to create a secant line whose slope approximates the derivative of H at $t = 2$. Write the equation of the secant line joining those two points. Why did you choose these two points?

$x_1 = 1$ $\quad$ $x_2 = 3$.

$m = \frac{0.5 - 2}{3 - 1} = -0.75$ $\quad$ $y_1 = 2$ $\quad$ $y_2 = 0.5$

$y = mx + c$.

$\Rightarrow 2 = -0.75 \times 1 + c$.

$\Rightarrow c = 2 + 0.75$

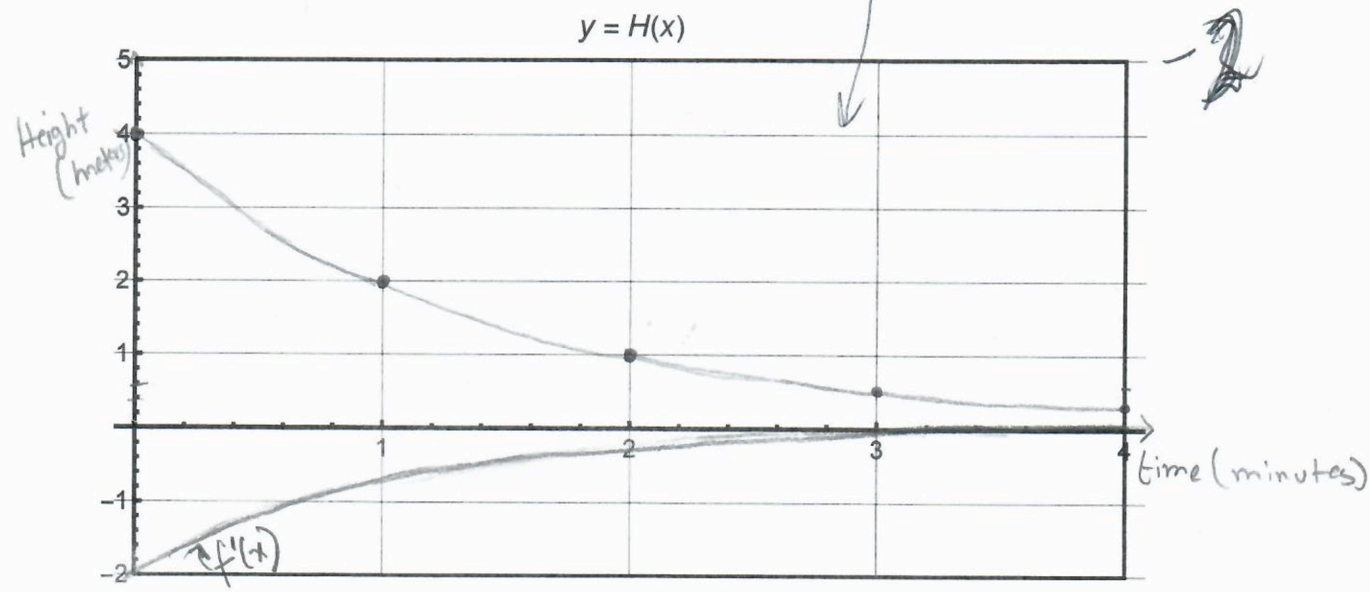
$\Rightarrow c = 2.75$

Secant line equation:

$\therefore \boxed{y = -0.75x + 2.75}$

I chose these two points as their gradient is -0.75, very close to -1. (grad at $\Rightarrow t = 2$)

c. (12 pts) Graph the data from the table, the secant line you used, and then your guess for the derivative function $H'(t)$.



My guess: After $t = 3$ the $f(x)$ curve flattens out, which is why my estimated $f'(x)$ graph follows x-axis after $x = 3$.

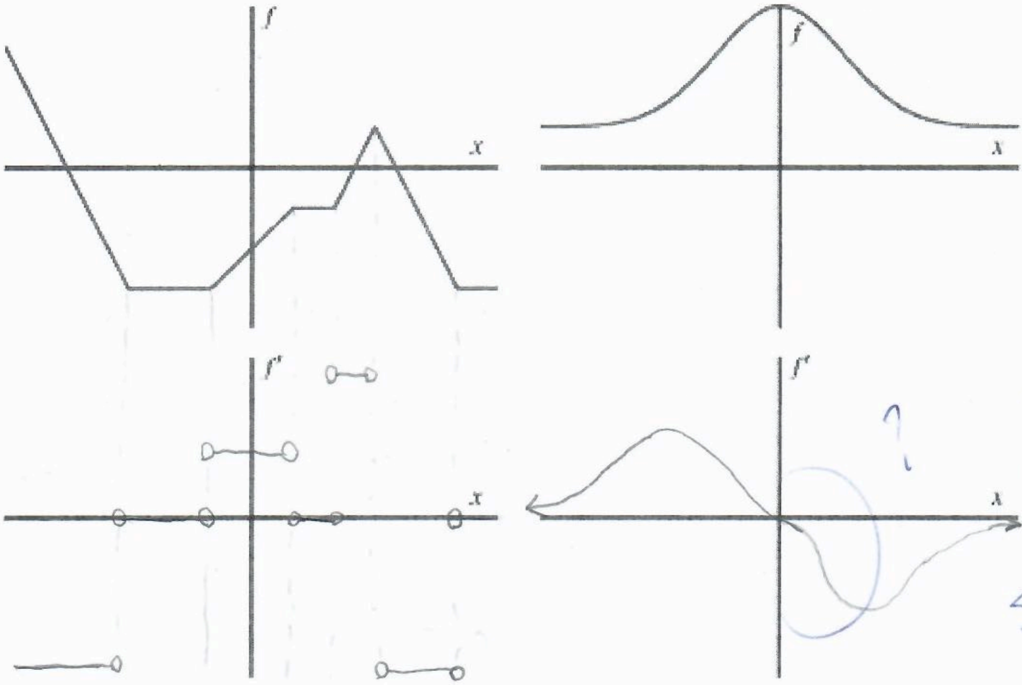
At $t < 3$ decreasing function, increasing negative gradient.

Well done

Problem 4 (25 pts)

a. (10 pts) For each graph of $y = f(x)$ in the following figures, your task is to sketch an approximate graph of its derivative function, $y = f'(x)$, on the axes immediately below it. View the scale of the grid for the graph of f as being 1*1, and assume the horizontal scale of the grid for the graph of f' is identical to that for f. If you need to adjust the vertical scale on the axes for the graph of f', you should label that accordingly.

excellent



Symmetric

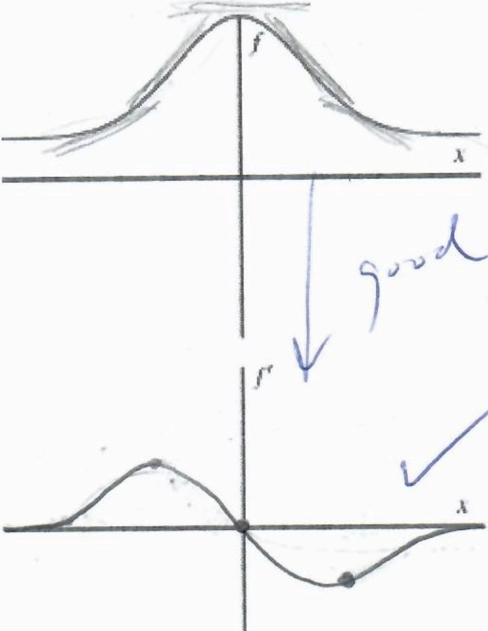
b. (7.5 pts) If the graph on the left represents the height of a particle, describe its motion; tell a story consistent with both the function and its derivative.

The partical is losing height and then stops. It then begins to rise at a slower speed than it was originaly going down before stopping again. After having stopped for a second time it begins rising at a greater speed than before. It suddenly begins going down at a speed comparable to the speed of its initial decent before coming to its third and final stop. Well done.

c. (7.5 pts) For the figure on the right, imagine that the graph describes your bank account over your lifespan. Explain what's going on, consistent with both the function and its derivative. Tell a story!

At age 18 you had some money in your bank account and got a job that just barely covered all of your living expenses. Over the years you were able to save more money after you got annual raises, while your cost of living stayed the same. Once you retire you begin to draw money from your account. For awhile you spend more money on things. At that rate you would quickly run out of money, so you begin to only buy essentials until you die with about the same amount of money in your account that you did when you were 18.

Nice work!



good – using slopes of tangent lines.