Directions: Each problem is worth 10 points. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Problem 1: Illustrate the chain rule by differentiating the following functions. Show your choices for the compositions; show all your work.

a. $h(x) = \sin(x^2 - 2x + 1)$

$f(x) = \sin(x)$, $f'(x) = \cos(x)$

$g(x) = (x^2 - 2x + 1)$, $g'(x) = (2x - 2)$

$\boxed{h'(x) = \cos(x^2 - 2x + 1) \cdot (2x - 2)}$

b. $h(x) = (e^x + \cos(x))^3$

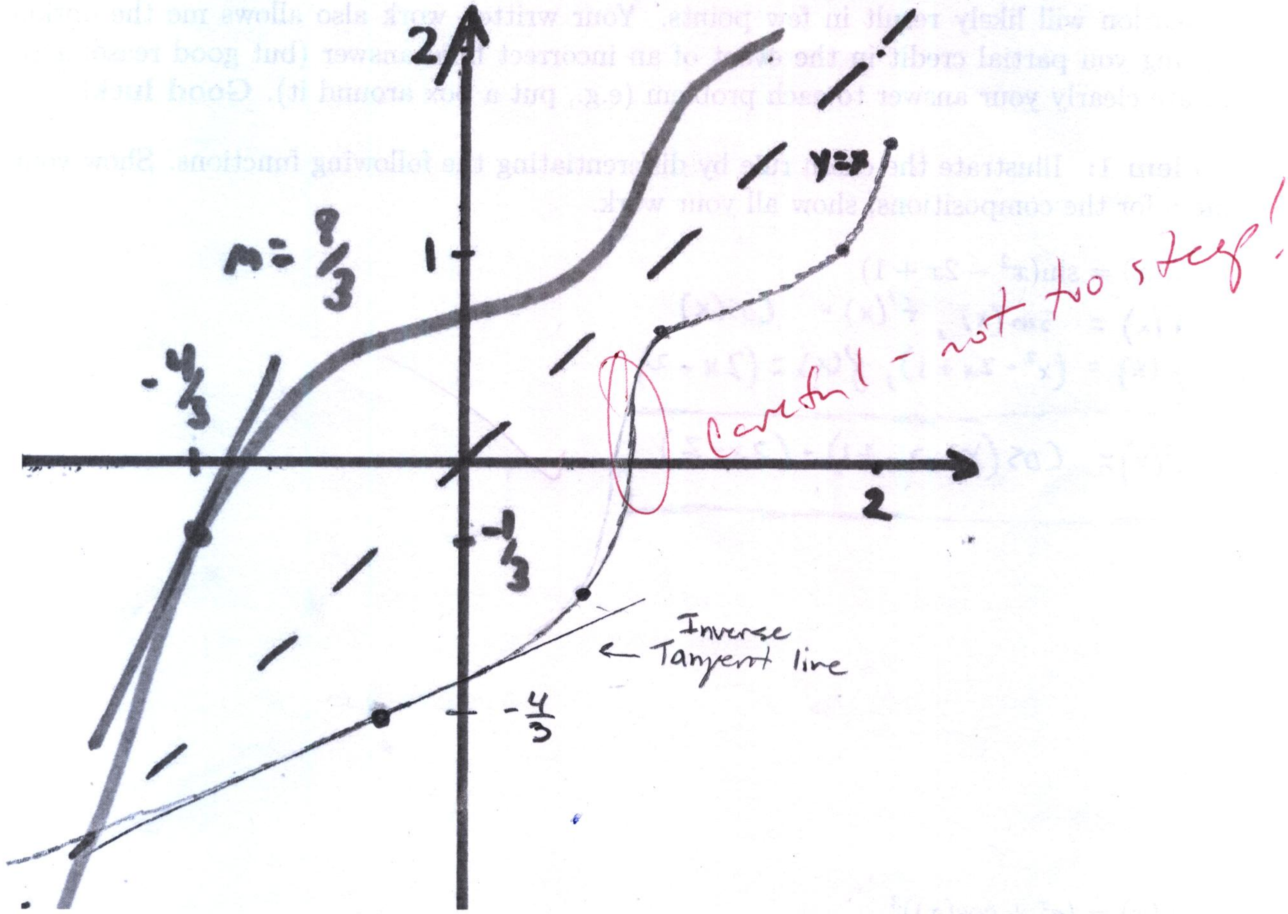
$f(x) = x^3$, $f'(x) = 3x^2$

$g(x) = (e^x + \cos(x))$, $g'(x) = (e^x - \sin(x))$

$\boxed{h'(x) = 3(e^x + \cos(x))^2 \cdot (e^x - \sin(x))}$

Problem 2: Inverse functions:

a. (4 pts) Draw (carefully!) the inverse function for the function f whose graph is given here:



b. (2 pts) Also draw (carefully!) the inverse of the tangent line shown in the figure.

c. (4 pts) Give the equations of both tangent lines.

i. The original tangent line:

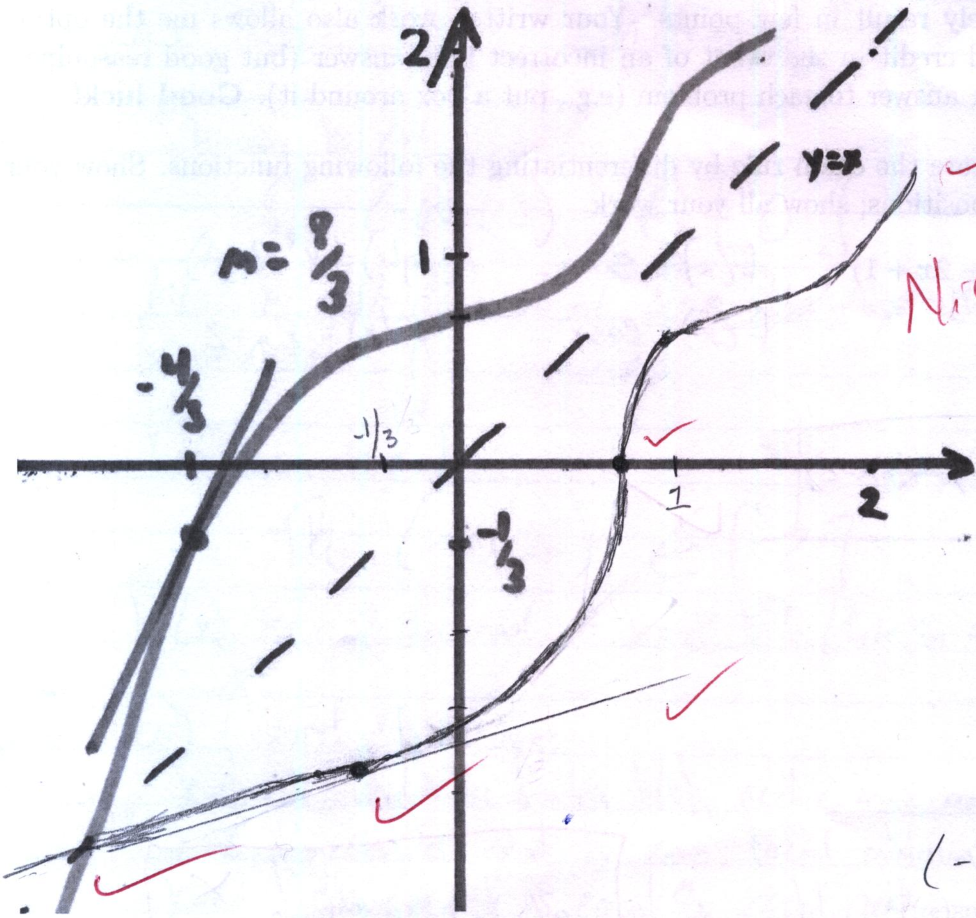
$y = \frac{8}{3}\left(x + \frac{4}{3}\right) - \frac{1}{3}$

ii. The tangent line for the inverse:

$y = \frac{3}{8}\left(x + \frac{1}{3}\right) - \frac{4}{3}$

Problem 2: Inverse functions:

a. (4 pts) Draw (carefully!) the inverse function for the function f whose graph is given here:



Nice graph

$(-\frac{4}{3}, -\frac{1}{3})$

$(-\frac{1}{3}, -\frac{4}{3})$

b. (2 pts) Also draw (carefully!) the inverse of the tangent line shown in the figure.

c. (4 pts) Give the equations of both tangent lines.

i. The original tangent line: $y = -\frac{1}{3} + \frac{8}{3}\left(x + \frac{4}{3}\right)$

$\underbrace{y_0 - m(x_0 + x)}_{?!!}$

$y = y_0 + m(x - x_0)$

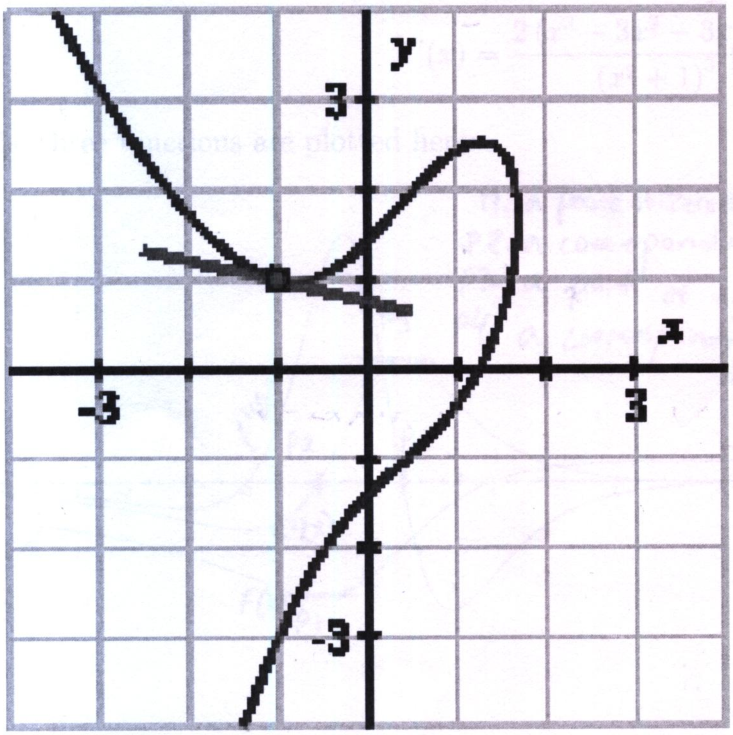
ii. The tangent line for the inverse:

$y = -\frac{4}{3} + \frac{3}{8}\left(x + \frac{1}{3}\right)$

Problem 3: For the curve given implicitly by

$$x^3 + y^2 - 2xy = 2,$$

shown below,



a. (6 pts) compute $y'(x)$ by implicit differentiation.

$$x^3 + y^2 - 2xy = 2$$

$$3x^2 + 2y \cdot y' - 2x \cdot y' + y \cdot 2 = 0$$

$$3x^2 + (2y \cdot y') - (2x \cdot y') + 2y = 0$$

Not sure what to do here

$$\rightarrow (2y \cdot y') - (2x \cdot y') = -3x^2 - 2y$$

$$y' = \frac{-3x^2 - 2y}{2xy}$$

factor out a y'

$$y'(2y - 2x) = -3x^2 + 2y$$

b. (4 pts) find the slope of the tangent line at (-1,1), and write the equation of the tangent line shown.

$$y' = \frac{-3x^2 - 2y}{-2xy}$$ ← I know the denom of this slope isn't right

$y' = -2.5$ ← so this is much larger than it should be

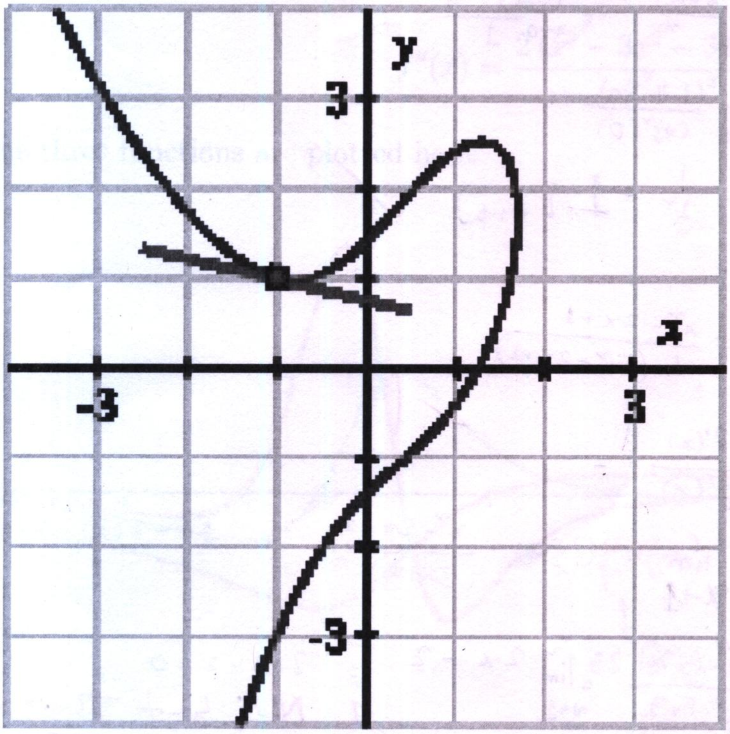
Good!

$$y = -2.5(x + 1) + 1$$

Problem 3: For the curve given implicitly by

$$x^3 + y^2 - 2xy = 2,$$

shown below,



a. (6 pts) compute $y'(x)$ by implicit differentiation.

$x^3+y^2-2xy=2$

$2x \quad y$
$2 \quad y'$

$\Rightarrow 3x^2+2yy'-2xy'-2y=0$

$\Rightarrow y'(2y-2x)=2y-3x^2$

$\Rightarrow y'(x) = \dfrac{2y-3x^2}{2y-2x}$

b. (4 pts) find the slope of the tangent line at (-1,1), and write the equation of the tangent line shown.

$x \quad y$

$\Rightarrow y'(x) = \dfrac{2\times 1 - 3(-1)^2}{(2\times 1)-(2\times -1)} = \dfrac{-1}{4} = -\dfrac{1}{4} := m$

$y = mx + c$ $\qquad \therefore y = mx + c$

$\Rightarrow c = y - mx = 1-(-1)\left(-\frac{1}{4}\right)$

$= \dfrac{3}{4}$

$\Rightarrow y = -\frac{1}{4}x + \frac{3}{4}$

Mason Milburn
Exam 3

Problem 4: Calculate the following limits (and show/explain your method!). Indicate clearly your final answers for each.

a. $\lim_{x\to 0} \frac{\tan(x)}{x}$

2 ↙ L'Hopital's rule

$\lim x \to 0 \ \frac{\sec(x)}{1} = \frac{1}{1} = \boxed{1}$

$\lim x \to 0 \ 1$

b. $\lim_{x\to 1} \frac{(x-1)^2}{(\ln((x-1)^2)}$

↙ is determinate

$\lim x \to 1 \ \frac{(1-1)^2}{(\ln((1-1)^2)} = \frac{0}{-\infty} = 0$

~

$\lim x \to 1 \ 0$

c. $\lim_{x\to\infty} \frac{\arctan(x)}{x}$

↙ is determinate because of arctan's limited domain

$\lim x \to \infty \ \frac{\arctan(\infty)}{\infty} = \frac{\frac{\pi}{2}}{\infty} = 0$

$\lim x \to \infty \ 0$

d. $\lim_{x\to 0^+} x\ln(x+1)$

↙ is determinate

$\lim x \to 0^+ \ 0 \cdot \ln(0+1) = 0 \cdot 0 = 0$

$\lim x \to 0^+ \ 0$

Problem 5: Consider the function $f(x) = \dfrac{x-1}{x^2+1}$.

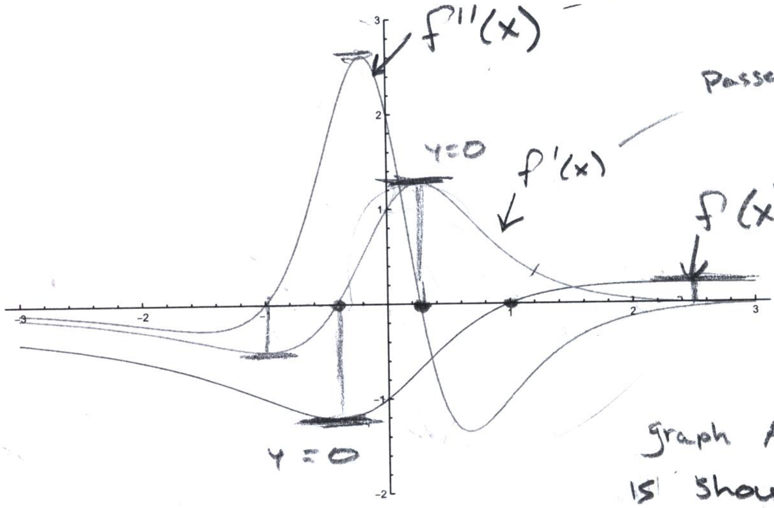
Its first derivative is

$$f'(x) = \frac{-x^2+2x+1}{(x^2+1)^2}$$

and the second derivative is

$$f''(x) = \frac{2(x^3-3x^2-3x+1)}{(x^2+1)^3}$$

The three functions are plotted here:



$f''(x)$ – Passes through the points $x=-1$ and $x=.23$ showing concavity.

Passes through $x=-.4$ and $x=2.5$ approximately

New slopes of $y=0$ tangent lines $x=-1$ and $x=.23$

$f(x)$ – x-intercept at $x=1$

Slopes of tangent lines at $y=0$ are $x=-.4$ and $x=2.5$ about so.

c.) From the left of the graph A decrease from all 3 graphs is shown and then an increase begins. The second derivative is showing it concave down, then up, then back down again and then an increase.

a. (3 pts) Identify which function is which (with reasons) on the graph.

b. (5 pts) Using the table below, indicate

i. the zeros and signs of the functions f, f', and f''. (If you can't find the **actual** values of the zeros, you can see and name them on the graph.)

ii. Asymptote information.

horizontal asymptotes

x	$-\infty$		-1		0		1		∞	
$f(x)$	$-$	$-.4$	$+$	0	$+$	$.23$	$+$	1	~~∞~~	
$f'(x)$	$+$	-1	$+$	$-.4$	$+$	$.23$	$-$	1	$-$	∞
$f''(x)$	$-$	$+$		$-$		$+$		$+$		

2

zeros?

c. (2 pts) To the right of the graph above, explain how the information in the table is consistent with the graph of f. You might annotate the graphs.

Problem 5: Consider the function $f(x) = \dfrac{x-1}{x^2+1}$.

Its first derivative is

$$f'(x) = \frac{-x^2+2x+1}{(x^2+1)^2}$$

$\dfrac{-2 \pm \sqrt{4+4}}{-2} = 1 \pm \sqrt{2} \approx \{2.4, -.4\}$

and the second derivative is

$$f''(x) = \frac{2\,(x^3 - 3x^2 - 3x + 1)}{(x^2+1)^3} = \frac{2(x+1)(x^2-4x+1)}{(x^2+1)^3}$$

$\dfrac{4 \pm \sqrt{16-4}}{2}$

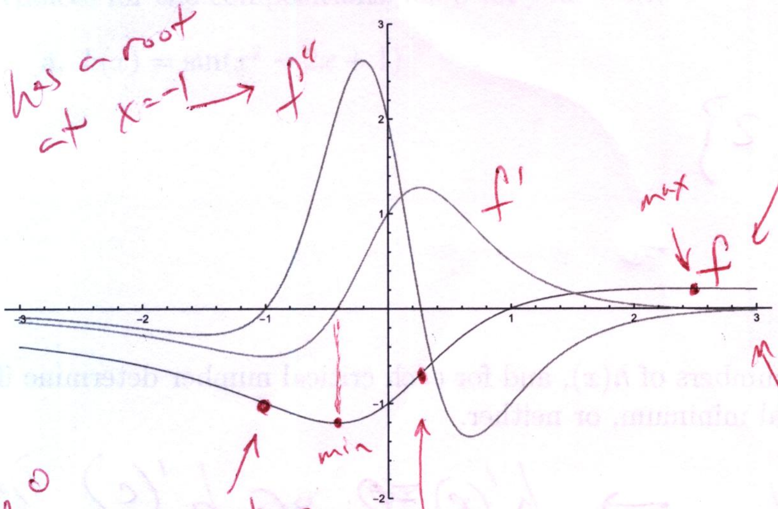
$2 \pm \sqrt{3}$

$\approx \{0.3, 3.7\}$

The three functions are plotted here:

has a root at $x=-1$ → f''

has a root at $x=1$



all heading to 0

inflection pt

all heading to 0

a. (3 pts) Identify which function is which (with reasons) on the graph.

b. (5 pts) Using the table below, indicate

i. the zeros and signs of the functions f, f', and f''. (If you can't find the **actual** values of the zeros, you can see and name them on the graph.)

ii. Asymptote information.

x	$-\infty$	-1	$1-\sqrt{2}$	0	$2-\sqrt{3}$	1	$1+\sqrt{2}$	$2+\sqrt{3}$	∞
$f(x)$	0	-1	~ -1.2	-1	$\sim -.7$	0	~ 0.3		0
$f'(x)$	0	$-$	0	1	$+$		0	$-$	0
$f''(x)$	0	0	$+$	2	0	$-$		0	0

c. (2 pts) To the right of the graph above, explain how the information in the table is consistent with the graph of f. You might annotate the graphs.

Problem 6: Let $h(x) = \dfrac{10(x-2)^{2/3}}{x}$. A computer algebra system gives the derivative as $h'(x) = \dfrac{10(6-x)}{3\sqrt[3]{x^2(x-2)}}$.

a. (2 pts) What are the domains of

i. $h(x)$:

$(-\infty, 0] \cup [0, \infty)$

$(-\infty, 0) \cup (0, \infty)$

-.5

ii. $h'(x)$:

$(-\infty, 0] \cup [0, 2] \cup [2, \infty)$

$(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

Rats, an interesting one, since it's not in the domain of h ...

b. (4 pts) Find all the critical numbers of $h(x)$, and for each critical number determine if it is a local maximum, a local minimum, or neither.

Critical numbers = $x=0$ $x=2$ $x=6$

-1

$h(x)$ local maximum = $x=6$

$h(x)$ local minimum = $x=2$

Justify!

Vertical Asymptote = $x=0$ ✓

good

c. (4 pts) What are the intervals of increase and the intervals of decrease of $h(x)$?

Decrease = $(-\infty, 0] \cup [0, 2] \cup [6, \infty)$

$(-\infty, 0) \cup (0, 2]$

Increase = $[2, 6]$ ✓

Problem 6: Let $h(x) = \dfrac{10(x-2)^{2/3}}{x}$. A computer algebra system gives the derivative as $h'(x) = \dfrac{10(6-x)}{3\sqrt[3]{x^2(x-2)}}$.

a. (2 pts) What are the domains of

i. $h(x)$: $\mathbb{R} - \{0\}$

ii. $h'(x)$: $\mathbb{R} - \{0, 2\}$

b. (4 pts) Find all the critical numbers of $h(x)$, and for each critical number determine if it is a local maximum, a local minimum, or neither.

Critical number $\to$ $h'(c) = 0$ or $h'(c)$ DNE

$h'(2)$ DNE, $h'(6) = 0$ $\Big\}$ critical numbers in the domain of h are $c \in \{2, 6\}$

h' changes sign as $x \to 2$, from $-$ to $+$: $\searrow \nearrow$

" " " $x \to 6$, from $+$ to $-$ $\nearrow \searrow$

min at $x = 2$, max at 6 (both local)

c. (4 pts) What are the intervals of increase and the intervals of decrease of $h(x)$?

Look at the signs of $h'(x)$

x		0		2		6	
$h'(x)$	$-$	$\Vert$	$-$	$\Vert$	$+$	0	$-$
$h(x)$	0 $\searrow$	$\Vert$	$\searrow$	0	$\nearrow$	$\frac{10\cdot 16^{1/3}}{6}$	$\searrow$ 0