Day 18, Mat128: Calculus A

Last Time Next Time

- Announcements:
  - If you need to Zoom to class, here's the link.
  - Your exams are graded.
    1. A key
    2. There was a built-in curve (the exam was out of 105 points); so we'll just consider your score as a percentage (out of 100).

- Comments/Reflections on some of your exams. There were a few things I want to emphasize, which we'll do by examining the key.
  1. Some of you didn't distinguish between F and f. You really should have used the same in the limit definition.
     The example in the key is just fine.
  2. On problem 2
     a. there were two approaches: both give the right answer, but one's simpler. Expand first, then differentiate the polynomial requires fewer steps.
     b. too many of you failed to treat xe^x as a product, and do the differentiation properly.
     c. This one was pretty darned straightforward. Algebra caused a few problems for a few of you, however.
  3. On problem 3, let's talk about part d, which concerns the tangent line.
  4. The third part of problem 4 caused trouble. We're using the linearization to estimate, and it should give a good result near to $x = 8$ (getting worse as we get farther from that point). Because of the concavity of the graph, we can conclude that our estimate is an over-estimate.
  5. I would like **well-labelled** graphs (and you'll recall that I asked you to memorize the sine function, so that you could reproduce it -- with important points indicated. So I would have liked to see the
     a. maximum and minimum values
     b. *x*-values for the extrema and inflection points;
     c. indication of the symmetry and periodicity

- Today's new topic: we're on to Section 2.7: Implicit Differentiation

  We'll approach this topic via some examples:

  - Consider the relationship between *x* and *y* given by

    $$y(x) = \frac{1}{x}$$

    We know how to differentiate *y(x)* with respect to *x*, using the power rule.

    But there's another way to think about this relationship, and that's

    $$xy = 1$$

    This gives no priority to either variable -- it's symmetric in both.

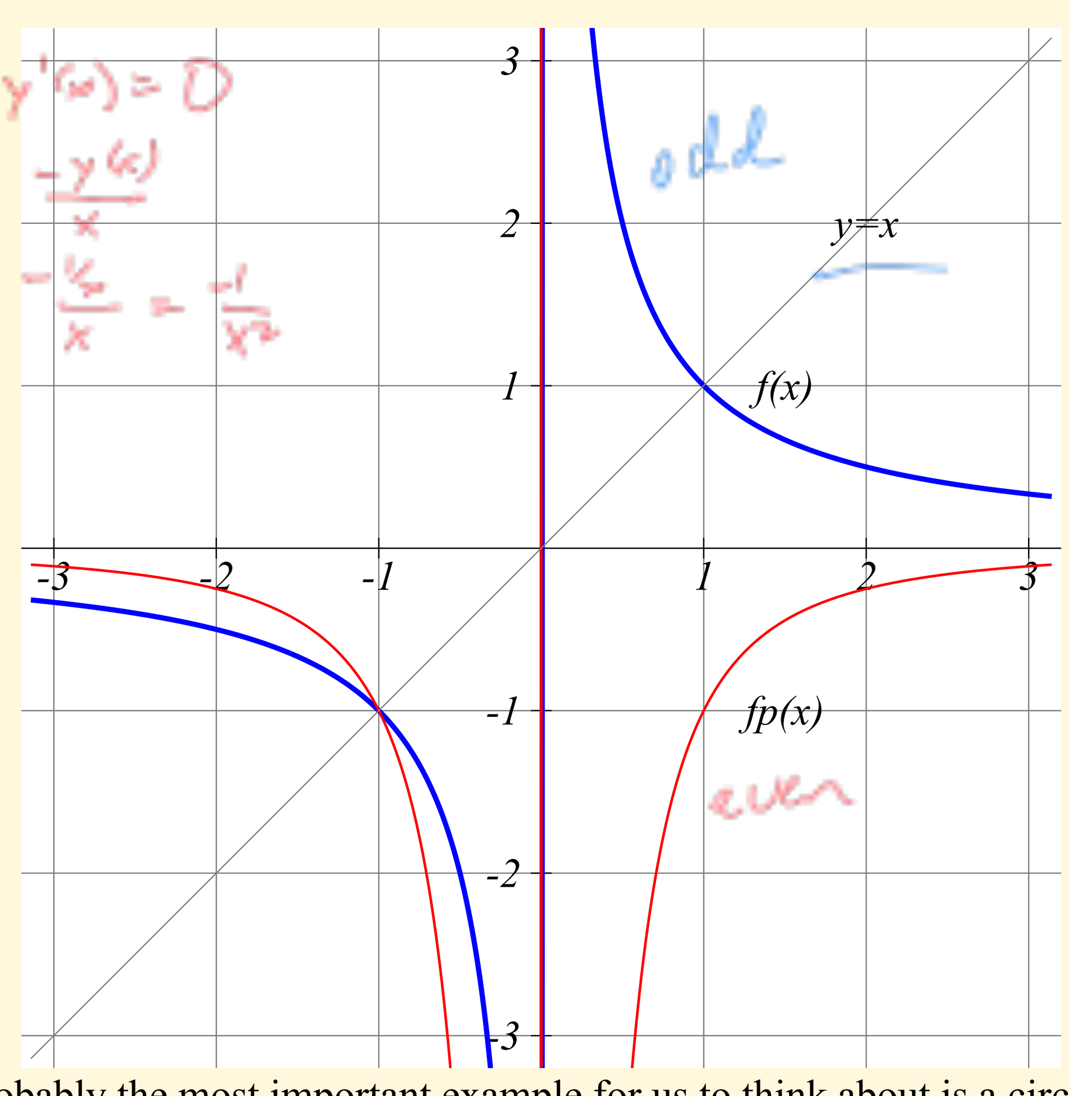
    (This means that the graph of y is symmetric about the line $x = y$ -- and that **could** be used to find the derivative of y from the derivative of x.)

    We can still differentiate to find *y'(x)*, however, using the product rule and something called "implicit differentiation". We consider y and unknown (implicitly defined) function of x, treating it as $y(x)$; then, since both sides of the equation

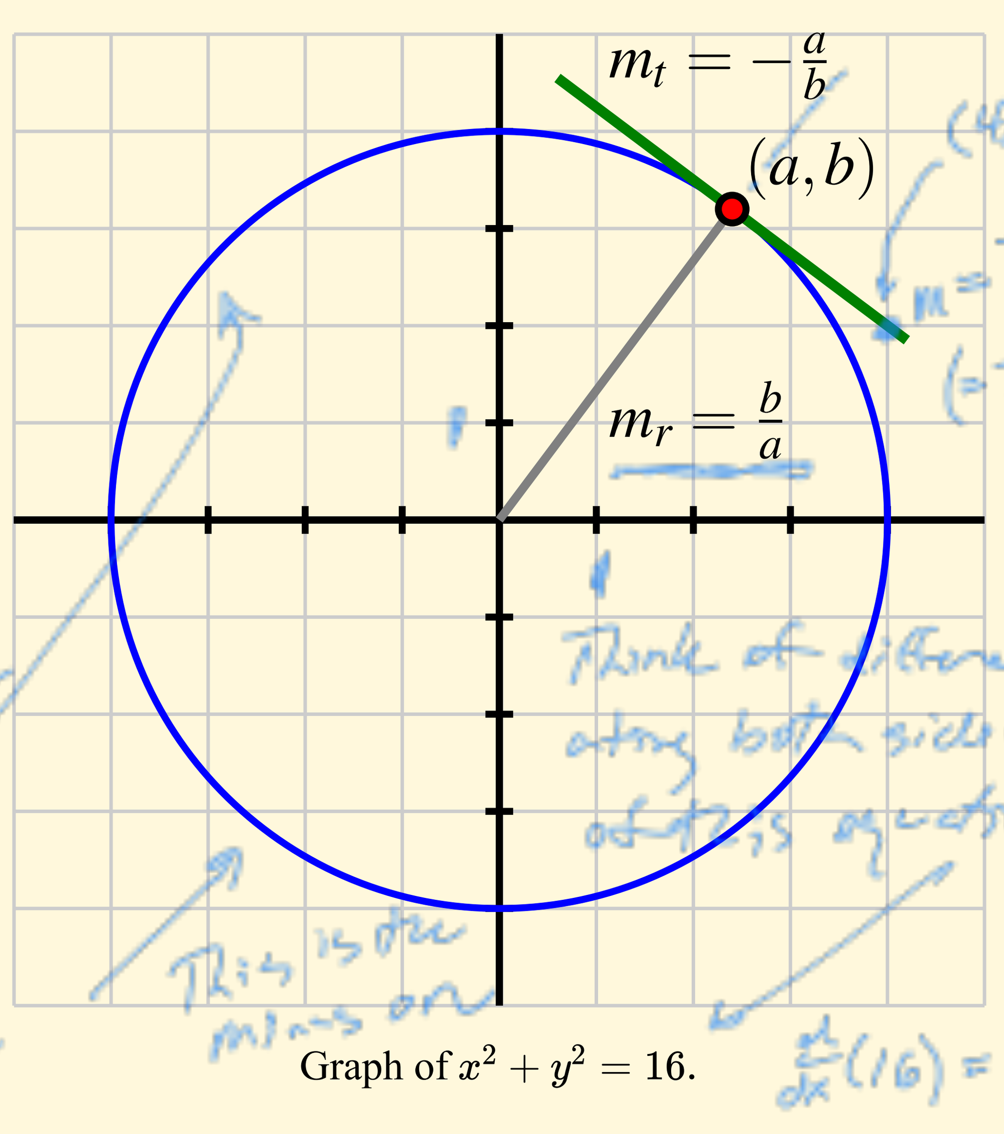
    $$xy(x) = 1$$

    are equal, the derivatives of both sides must be equal. We differentiate both sides, and equate them (using the product rule on the left). From this we obtain the correct derivative, as well.

    The solution curves are hyperbolas (one of the "conic sections": those curves you get by slicing a cone).



  - Probably the most important example for us to think about is a circle:



Graph of $x^2 + y^2 = 16$.

    One of the issues is that this is not the graph of a function -- it fails the vertical line test. But it is clearly a really important graph for us to be able to do calculus on (e.g. make tangent lines).

    One approach to dealing with this object is to divide it into two separate functions: an upper and a lower function, which would look like this:

    $$y = \pm\sqrt{16 - x^2}$$

    The upper function corresponds to the "+" sign, whereas the lower corresponds to the "-" sign. But what a pain! This is where dealing with it as an implicit function helps us.

    As an example of a problem one might want to solve for the graph above, what are the values of a and b such that the tangent to the circle goes through the point (4,2)?

    I can think of two ways to solve this:

    1. Using geometry and symmetry (and just one equation); or
    2. Using **two** equations: the equation of the circle, and the equation of the tangent line.

  - Another example is the **folium of Descartes**, which is the graph (solution) of the implicit equation but

    $$x^3 + y^3 = 6xy$$

    Here the equation is perfectly symmetric in x and y: it's not at all clear which variable is dependent, and which independent. By the way, this **symmetry** in x and y means that the solution curve must be **symmetric about the line** $y = x$:

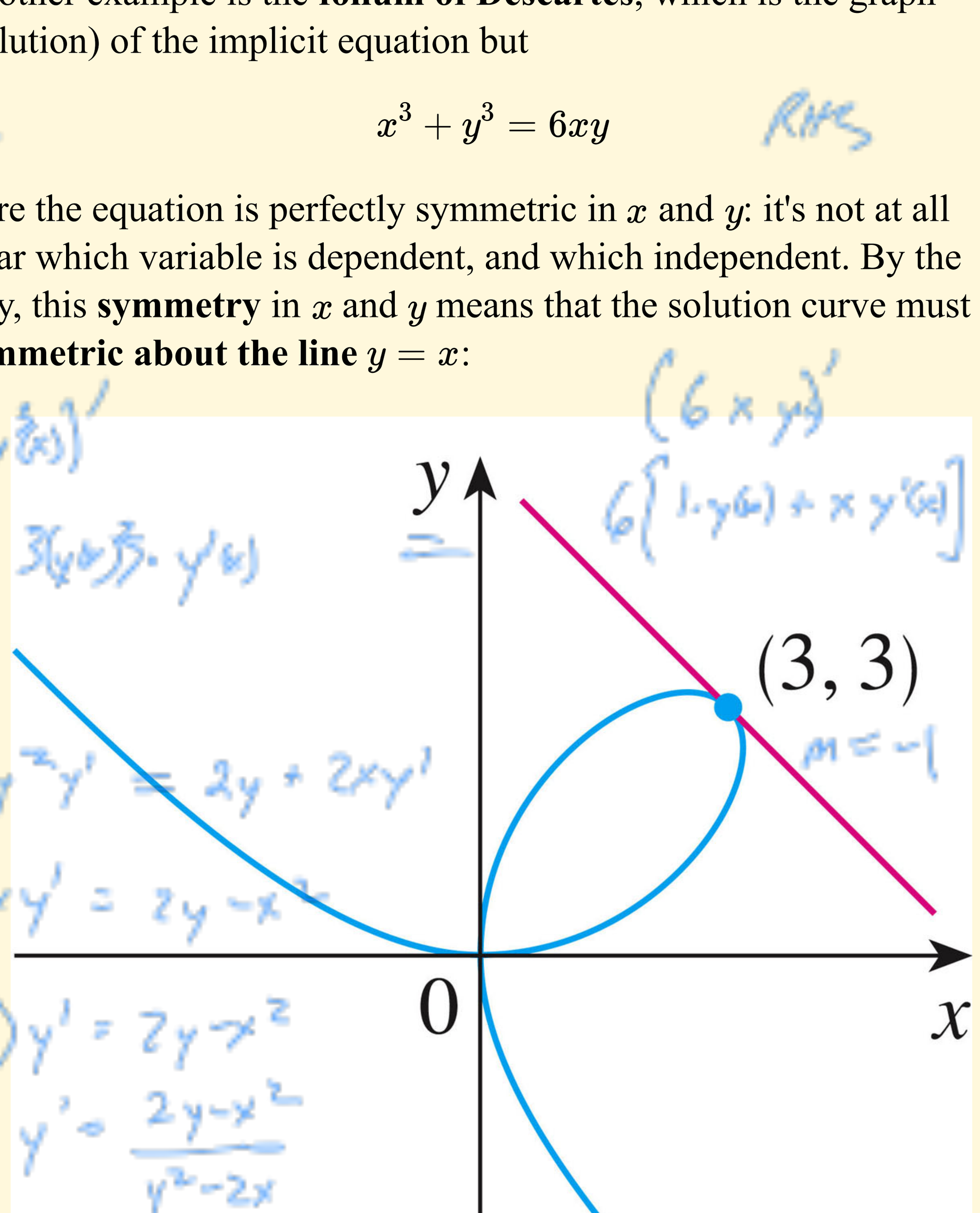


FIGURE 4

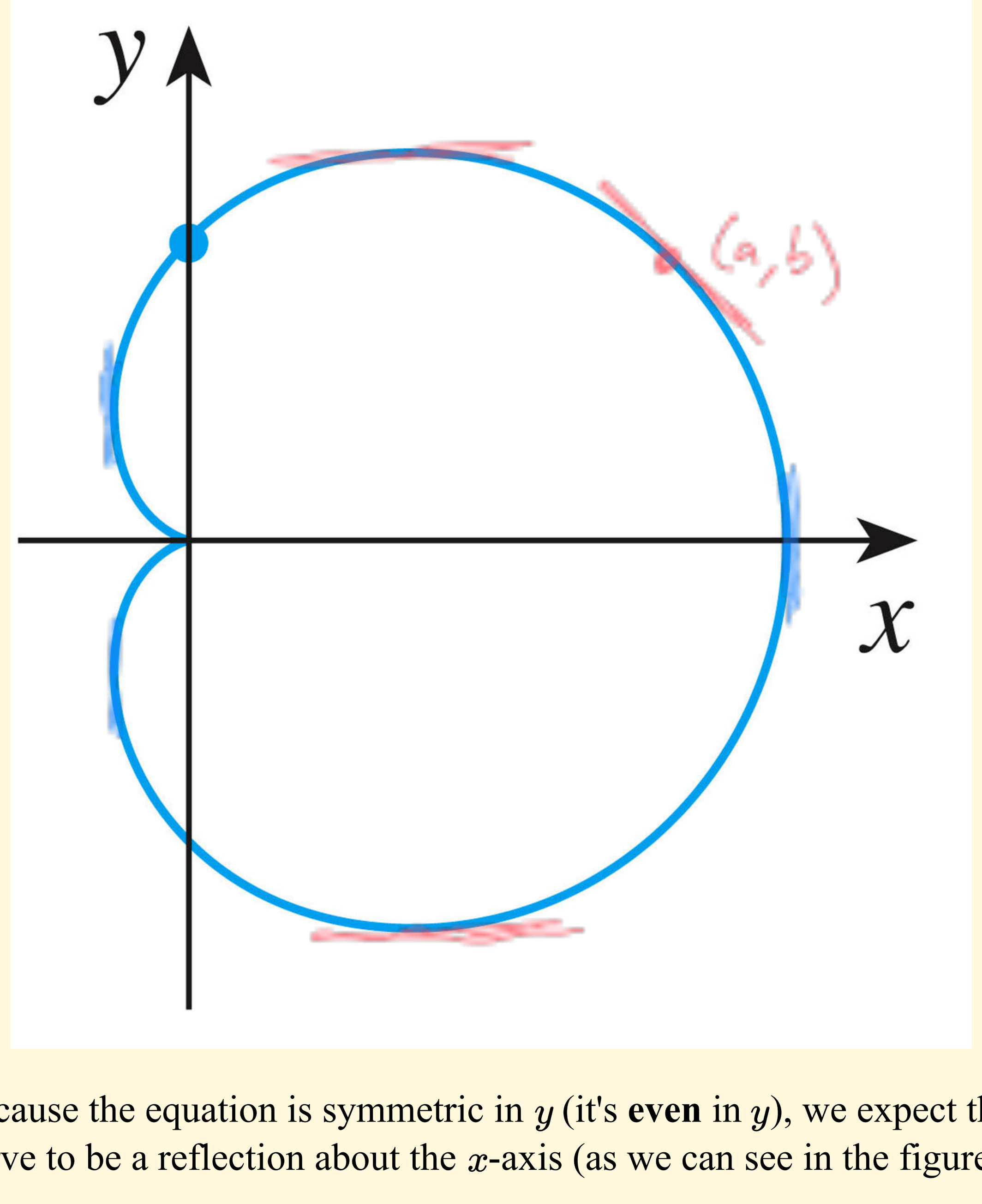
    Q: What is the equation of the tangent line shown?

    Even though the graph is not the graph of a function (either $y(x)$ or $x(y)$), we can still find tangent lines, etc., using ordinary derivatives. It is the case, however, that a single point in the plane can be represented by several different tangent lines (e.g. the point (0,0))

    Q: What are the **two** equations of the tangent line to the point (0,0)?

  - Here's a more complicated relationship between *x* and *y* given by

    $$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$



    Because the equation is symmetric in y (it's **even** in y), we expect the curve to be a reflection about the x-axis (as we can see in the figure).

    So if (x, y) is a point on the graph, then $(x, -y)$ will also be on the graph (and *vice versa*).

    $$x^2 + y^2 = x^2 + (-y)^2 = (2x^2 + 2(-y)^2 - x)^2 = (2x^2 + 2y^2 - x)^2$$

  - Here's a worksheet for us to work on (it includes a few chain rule problems, too: from now on we'll be bumping into more compositions, and compositions shout "chain rule!").

- **Links**:
  - A Chain Rule tutorial

Website maintained by Andy Long. Comments appreciated.
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