Section 2.8: using derivatives to Evaluate limits

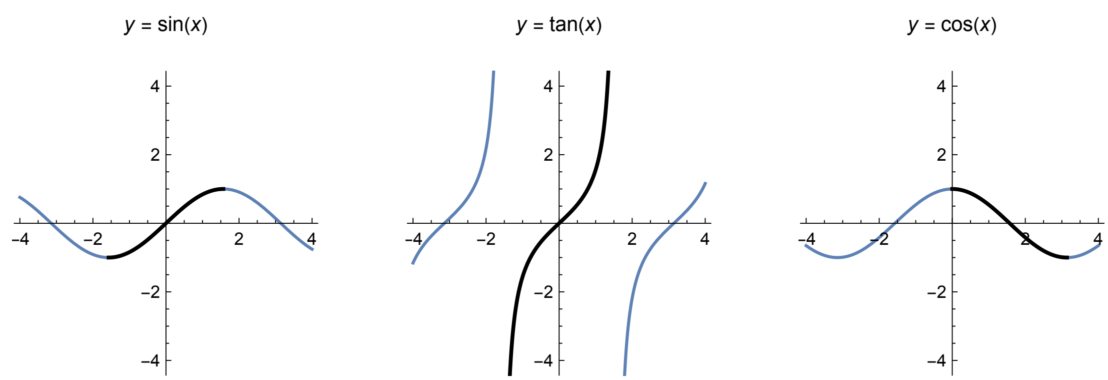
Supporting materials

- Boelkins/Austin/Schlicker's Active Calculus

Review

Trigonometric functions, and their inverses:

Out[95]=



Questions

- What is the domain and range of $\sin^{-1}(x)$? D: [-1,1] R: [-Pi/2,Pi/2]
- Of $\tan^{-1}(x)$? D: all reals; R: (-Pi/2,Pi/2)
- Of $\cos^{-1}(x)$? D: [-1,1]; R: [0,Pi]

- When is an "identity" an identity?

In[96]:=
```
Tan[ArcTan[0.3]]
ArcTan[Tan[Pi / 4 + 2 Pi]]
ArcCos[Cos[Pi / 5 + 2 Pi]]
{N[%], N[Pi / 5]}
ArcCos[Cos[-Pi / 5]]
{N[%], N[-Pi / 5]}
Sin[ArcCos[1 / 2]]
```

Out[96]= 0.3

Out[97]= $\frac{\pi}{4}$

Out[98]= $\text{ArcCos}\left[\frac{1}{4}\left(1+\sqrt{5}\right)\right]$

Out[99]= {0.628319, 0.628319}

Out[100]= $\text{ArcCos}\left[\frac{1}{4}\left(1+\sqrt{5}\right)\right]$

Out[101]= {0.628319, -0.628319}

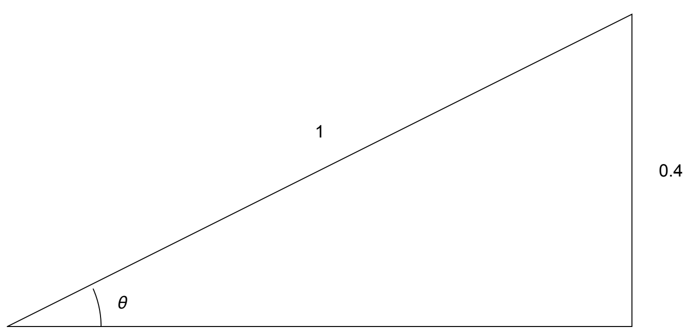
Out[102]= $\frac{\sqrt{3}}{2}$

- What is $\tan\left(\tan^{-1}(0.3)\right)$? 0, 3
- What is $\tan^{-1}(\tan(\pi/4+2\,\pi))$? $\pi/4$
- What is $\cos^{-1}(\cos(\pi/5))$? $\pi/5$
- What is $\cos^{-1}(\cos(-\pi/5))$? $\pi/5$
- What is $\sin\left(\cos^{-1}(1/2)\right)$? $\sqrt{3}/2$

Right triangles

To find $\tan\left(\sin^{-1}(0.4)\right)$, let $\theta=\sin^{-1}(0.4)$ so that $0.4=\sin(\theta)$. Represent $\sin(\theta)$ in a right triangle.

Out[103]=



Using this triangle, $\tan\left(\sin^{-1}(0.4)\right)=\tan(\theta)=\frac{0.4}{\sqrt{1-0.4^2}}=\frac{0.4}{\sqrt{0.84}}\approx 0.436$.

Indeterminate forms

End behavior of functions requires that we deal with limits. An "end" may occur where $x \to \pm\infty$, or it may occur internally -- for instance, where there is a vertical asymptote.

Questions

- What is $\lim_{x\to 0} \frac{2x+1}{x-1}$? Determinate; Plug in!

```
In[104]:= Limit[(2 x + 1) / (x - 1), x → 0]
Out[104]= -1
```

- What is $\lim_{x\to 0} \frac{4x+1}{x}$? $x=0$; $-\infty \,\|\, \infty$

```
In[105]:= Limit[(4 x + 1) / x, x → 0]
Out[105]= Indeterminate
```

- What is $\lim_{x\to 0} \frac{2x}{x}$? $\frac{2x}{x} = 2$ [cancel!]

```
In[106]:= Limit[2 x / x, x → 0]
Out[106]= 2
```

- What is $\lim_{x\to 0} \frac{x}{5x}$? $\frac{x}{5x} = \frac{1}{5}$ [cancel!]

```
In[107]:= Limit[x / (5 x), x → 0]
Out[107]= 1/5
```

- What is $\lim_{x\to \infty} \frac{x}{2}$? ∞

```
In[108]:= Limit[x / 2, x → Infinity]
Out[108]= ∞
```

determinate

- What is $\lim_{x\to \infty} \frac{1}{x}$? 0

```
In[109]:= Limit[1 / x, x → Infinity]
Out[109]= 0
```

- What is $\lim_{x\to -\infty} \frac{x-1}{x+1}$? Indeterminate: $\frac{\infty}{\infty}$

Use L'Hôpital; or rewrite (use algebra!)

```
In[110]:= Limit[(x - 1) / (x + 1), x → -Infinity]
Out[110]= 1
```

$$\frac{x-1}{x+1} = \frac{x+1-1-1}{x+1} = 1 - \frac{2}{x+1}$$

- What is $\lim_{x\to \infty} \frac{x^2}{x}$? Cancel

```
In[111]:= Limit[x^2 / x, x → Infinity]
Out[111]= ∞
```

Given a limit $\lim_{x\to a} f(x)$, if we can simply evaluate $f(a)$ as the limit we say $\lim_{x\to a} f(x)$ is *determinate*. If we

determinate.

cannot simply evaluate $f(x)$ at $x = a$, we say the limit is *indeterminate* -- it may or may not exist. Perhaps some algebra will help!

Indeterminate limits

- $\frac{0}{0}$, a small number divided by a small number -- hmm, could be anything. More work is needed.
- $\frac{\infty}{\infty}$, a large number divided by a large number could be anything. More work is needed.

The most important indeterminate form in calculus is undoubtedly the limit definition of the derivative, in either of its forms:

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

or

$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

In the limit as $h \to 0$, the numerator goes to 0 and the denominator goes to 0; similarly as $x \to a$ in the second form of the limit definition.

L'Hôpital's rule

If you have a limit of a quotient which is either a $\frac{0}{0}$ or an $\frac{\infty}{\infty}$ limit, then the following is true if the limit (and the derivatives) exists:

$$\lim_{x\to a} \frac{g(x)}{h(x)} = \lim_{x\to a} \frac{g'(x)}{h'(x)}$$

Warning: If the given limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then the above two limits are not equal.

Example

To evaluate $\lim_{x\to\infty} \frac{x}{e^x}$, first notice that plugging in ∞ for x produces $\frac{\infty}{e^\infty} = \frac{\infty}{\infty}$. We can use L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{(x)'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Questions

Evaluate the following limits.

- $\lim_{x\to\infty} \frac{x}{\ln(x)}$ $= \frac{\infty}{\infty}$ $\lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$

In[112]:= `Limit[x / Log[x], x → Infinity]`

Out[112]= ∞

- $\lim_{x\to\infty} \frac{\tan^{-1}(x)}{x^2+1}$ $= 0$ determinate arctan(x) $\to \frac{\pi}{2}$ as $x \to \infty$

In[113]:= `Limit[(ArcTan[x]) / (x^2 + 1), x → Infinity]`

Out[113]= 0

- $\lim_{x\to 1} \frac{x^2-3x+2}{\sin(x-1)}$ $= \lim_{x\to 1} \frac{2x-3}{\cos(x-1)} = -1$

In[114]:= `Limit[(x^2 - 3 x + 2) / (Sin[x - 1]), x → 1]`

Out[114]= -1

- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$ $= \lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2}$

In[115]:= `Limit[(1 - Cos[x]) / x^2, x → 0]`

Out[115]= $\frac{1}{2}$

Why it works

For the $\frac{0}{0}$ case, this means $f(a) = 0$ and $g(a) = 0$. Remember the limit definition of the derivative

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$$

Since $f(a) = 0 = g(a)$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$
$$= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$
$$= \frac{f'(a)}{g'(a)}$$

Questions

Can we use L'Hôpital's rule on $\lim_{x\to 1}\frac{x-1}{e^{x-1}}$? Compare the actual value of this limit with the limit that comes from L'Hôpital's rule.

No: not indeterminate

In[116]:=
```
Limit[(x - 1) / E^(x - 1), x → 1]
Limit[1 / E^(x - 1), x → 1]
```

Out[116]= 0

Out[117]= 1

Other indeterminate forms

There are other indeterminate forms. Sometimes we can use L'Hôpital's rule to evaluate them *if* we can rewrite into either the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

Other forms

- $\infty - \infty$ or a large number minus a large number.
- $0 \cdot \infty$ or a number close to zero times a large number.
- Indeterminate powers
  - 0^0 or a small number raised to another small number.
  - ∞^0 or a large number raised to a small number.
  - 1^∞ or a number close to 1 raised to a large power.

Product example

Evaluate $\lim_{x\to\infty} x(\pi/2 - \tan^{-1}(x))$.

Rewrite one of the factors as a fraction, factor = $\frac{1}{1/\text{factor}}$.

$\lim_{x\to\infty} x(\pi/2 - \tan^{-1}(x)) = \lim_{x\to\infty}\frac{(\pi/2-\tan^{-1}(x))}{1/x} = ?$ $\frac{0}{0}$

Use l'Hôpital:

$\lim_{x\to\infty}\frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} =$

In[118]:=
```
Limit[x (Pi / 2 - ArcTan[x]), x → Infinity]
```

Out[118]= 1

$\lim_{x}\frac{x^2}{1+x^2} = \lim_{x\to\infty}\frac{2x}{2x}$

Questions

- $\lim_{x\to\infty} x \sin\left(\frac{2}{x}\right)$

$$= \lim_{x\to\infty} \frac{\sin(2/x)}{1/x} = \lim_{x\to\infty} \frac{\cos(2/x)\left(-\frac{2}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} 2\cos(2/x) = 2$$

In[119]:= `Limit[x Sin[2 / x], x → Infinity]`

Out[119]= 2

- $\lim_{x\to 0^+} x \ln(x)$

In[120]:= `Limit[x Log[x], x → 0, Direction → "FromAbove"]`

Out[120]= 0

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x} -x = 0$$